Radial basis functions collocation method for solution of MagnetohydrodynamicNanoboundary-layer flows

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ABSTRACT
In this paper, axisymmetric flows over stretching surfaces in a porous medium in the presence of a magnetic field with the Navier boundary condition have been considered. The governing partial differential equation (PDE) has been reduced to nonlinear ordinary differential equation (ODE) by using suitable similarity transformations. The collocation methods based on Multiquadric (MQ) Radial basis function (RBF) have been developed for the arising ODE. The method has been tested on an example to show that the presented method is easy applicable and simple. The computed solution is in a good of the solved example with the exact solution. This has been verified by the plotted graphs.

Keywords: Collocation; MQ; Nano boundary-layer flows; Radial basis function; method.

INTRODUCTION
The notion of a boundary layer was first introduced by (Prandtl, 1904) to explain the discrepancies between the theory of inviscid fluid flow and experiment. In the classical boundary-layer theory, the condition of no-slip near the solid walls is usually applied, where the fluid velocity component is assumed to be zero relative to the solid boundary that is not true for fluid flows at the micro- and Nano scale. (Gad-el-Hak, M, 1999) showed that the no-slip condition is no longer valid instead, a certain degree of tangential slip must be allowed. The Nano boundary-layer fluid flows mean Nano scale flows which have many applications in micro-electro-mechanical systems. In recent years, because of importance of applications of fluid flow over a stretching sheet in industry, such as polymer processing of a chemical engineering plant and metallurgy, some interests have been given and some useful results have been introduced by many authors, such as (Aly. E. H., Ebaid, 2012, 2014), (Babolian. E and Hosseini. M, 2003, 2012), (Brown and et al, 2005), (Buhman. M and Dyn. N, 1993), (Chantasiriwan. S, 2004), (Darvishi and et al, 2008), (Gad-el-Hak. M, 1999), (Ghahsareh. HR and et al, 2012), (Ghovatmand and et al, 2011), (Matthews and et al, 2007, 2008), (Rashidi and et al, 2011), (sakiadis. B.C, 1961), (soltanalizadeh. B and et al, 2013), (Turkyilmazeylu. M, 2013), (He. J. H and et al, 2007) and (wang. C. Y, 2009). Continuously, moving surface's property of flow causes to use it in several thermal processes and moisture treatment of materials, predominantly in processes involving continuous pulling of a sheet through a reaction zone. For example in metallurgy, textiles and paper industries. Very recently, (Van Gorder and et al, 2010) presented similar solutions for the Nano
boundary-layer flows with Navier boundary condition. Existence and uniqueness solutions were established by (Akyildiz. F. T and et al. 2011). They considered the three dimensional nano boundary-layer viscous flows over a horizontal axisymmetric two-dimensional stretching surface. Let \( (u, \theta, w) \) be the velocity components along \((x, y, z)\) axes, respectively. The three dimensional nano boundary-layer flows of an incompressible fluid with constant electrical conductivity \( \sigma \) over a horizontal axisymmetric two dimensional stretching elastic sheet through the porous media is as follows:

\[
\begin{align*}
\rho_{\theta}(u_x + \theta_y + w_z & = 0, \tag{1} \\
\rho(u_{xx} + \theta_{xy} + w_{xz}) + p_x + \sigma \beta n \theta u + \frac{\sigma}{k} u &= \mu (u_{xx} + u_{yy} + u_{zz}) , \tag{2} \\
\rho(u_{yy} + \theta_{yx} + w_{yz}) + p_y + \sigma \beta n \theta w + \frac{\sigma}{k} w &= \mu (u_{xx} + u_{yy} + u_{zz}) , \tag{3} \\
\rho(u_{zz} + \theta_{zx} + w_{zy}) + p_z + \frac{\sigma}{k} w &= \mu (u_{xx} + u_{yy} + u_{zz}) , \tag{4}
\end{align*}
\]

with the boundary conditions:

\[
\begin{align*}
u_x &= 0, \quad \theta_y = (m - 1) \gamma f \phi, \quad w = 0, \quad \text{if } z = 0, \tag{5} \\
\text{and} \quad u_x &= 0, \quad \theta_y = 0, \quad w = 0, \quad \text{if } z = \infty.
\end{align*}
\]

Where \( \rho, p, B, \sigma, \mu \) and \( k \) are the fluid density, pressure, uniform transverse magnetic field of strength, electrical conductivity, viscosity and permeability of the porous medium and \( c > 0 \) is stretching constant and \( m \) is a parameter that describes the type of plate stretching respectively. We have two-dimensional stretching wall if \( m = 1 \), and for \( m = 2 \) we have axisymmetric stretching \( \theta = \frac{w}{m - 1} \) is the constant of suction/injection velocity at the wall. \( s < 0 \) when injection occurs from the surface and velocity \( s > 0 \) is due to suction velocity. The induced magnetic field is assumed negligible. Also external electric field is zero because of polarization of charges. According to (Wu. L. A’s article, 2008) \( U_{\text{slip}} \), slip velocity at the wall, can be presented as:

\[
U_{\text{slip}} = \left( \frac{2 - 2m^2}{9} \right) \lambda \phi u - \left( \frac{1}{4} + \frac{2}{3m^2} \right) \lambda \phi u_{yy} + \left( \frac{1}{4} + \frac{2}{3m^2} \right) \lambda \phi u_{yy}.
\]

(6)

Where \( 0 \leq \alpha < 1 \) is the momentum accommodation and \( 0 \leq \alpha = \frac{1}{\alpha}, 1 \leq 1 \).

The positive parameter \( \lambda \) is the dimensionless slip parameter. First of all we introduce the dimensionless function \( f(\phi) \) with similarity variable \( \phi \) as follow:

\[
u = c \sqrt{f(\phi)}, \quad s = (m - 1) \gamma f \phi, \quad w = m \sqrt{f(\phi)}, \quad \phi = \frac{m+1}{\sigma}.
\]

(7)

Where \( \sigma \) is the kinematic viscosity. By using Equation 7 with Equations 2-4 we can get a non-dimensional nonlinear ordinary differential equation governing the resulting boundary-layer flow, the Navier–Stokes equation as follows:

\[
f''''(\phi) + 6f(\phi)f''(\phi) - (f'(\phi))^2 - \left( \frac{M + \frac{2}{\alpha} \gamma}{\phi} \right)f'(\phi) = 0,
\]

(8)

With the following boundary conditions:

\[
f(0) = s, \quad f'(0) = 1 + \gamma f''(0) + \delta f'''(0), \quad f'(\infty) \to 0,
\]

(9)

Where \( M = \frac{2\beta^2}{\sigma} \) and \( k_p = \frac{\sigma}{\nu} \) are the magnetic and dimensionless permeability parameters. \( \gamma = \frac{\beta \lambda}{\sigma} \) and \( \delta = \frac{2 \lambda}{\sigma} \). denote the first order velocity slip parameter and the second order velocity slip parameter (see (VanGorder. R. A, 2010) for more details). We denote that \( \gamma = \frac{1}{\alpha}, \quad \phi \in (0, \infty) \). The solution of the Navier stock equation (8) with boundary conditions (9) has been considered by many authors. (Samadpoor. S et al.2013) investigated the model numerically and obtained the solution of the problem. In recent years many of numerical and analytical techniques are introduced and well-used for finding and approximating the similarity solution of

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This paper is organized in six sections. In section two, summery of RBFs in general have been given and in section three a mesh free method based on the RBFs has been applied to Equation (8) subject to its boundary conditions (9).

In section four, numerical results and discussions are brought and finally in section five and six, there are conclusion and references.

**RBF Introduction**

The method of RBF is a well-known family of meshless methods that is the center point is not necessarily structured.

These functions have been used for solving various kind of problems such as (Abbasbandy. S and et al, 2012-2014), (Buhmann. M and Dyn. N, 1993), (Brown. D. and et al, 2005), (chantasiriwan. S, 2004), (Michelli. CA,1986) and (Sarra. SA, 2005). In all the interpolation methods for scattered data sets, RBFs outperforms all the other methods regarding accuracy, stability, efficiency, memory requirement and simplicity of the \( \varphi(x, x_j) = \varphi(\eta_j) = \sqrt{\eta_j^2 + c_j^2} \), where \( \eta_j = ||x - x_j|| \) is the Euclidian norm.

The RBF may also have a shape parameter \( c \), in which case \( \varphi(\eta) \) is replaced by \( \varphi(\eta, c) \). Some of the most popular RBFs are given in Table.1.

**Table.1:** Some of the most popular radial basis functions.

<table>
<thead>
<tr>
<th>RBFs</th>
<th>Approximation</th>
<th>( r, c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infinitely smooth</td>
<td>( \varphi(r, c) )</td>
<td></td>
</tr>
<tr>
<td>Multiquadric (MQ)</td>
<td>( \sqrt{1+(\eta r)^2} )</td>
<td></td>
</tr>
<tr>
<td>Inverse multiquadric (IMQ)</td>
<td>( \sqrt{1+(\eta r)^2} )</td>
<td></td>
</tr>
<tr>
<td>Inverse quadric (IQ)</td>
<td>( \sqrt{1+(\eta r)^2} )</td>
<td></td>
</tr>
<tr>
<td>Gaussian(GS)</td>
<td>( \eta^\frac{1}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

The RBF interpolant is a linear combination of RBFs centered at the scattered points \( x_j \):

\[
\hat{f}(x) = \sum_{j=1}^{N} \lambda_j \varphi(c, \eta),
\]

(10)

When we use infinitely smooth RBFs, the approximations feature spectral convergence as the points get denser. This has been proven strictly only for some special cases, like (Buhman. M and Dyn. N’s article, 1993). We will see that the accuracy of the solution is a function of the shape parameter and that very small values of \( c \) often give the best results. The shape parameter affects both the accuracy of the approximation and the conditioning of the interpolation matrix (Sarra. SA, 2005). According to definition of \( \eta \) approximation of \( f^m(x, c) \) can be rewritten as follow respectively:

\[
\hat{f}(x) = \sum_{j=1}^{N} \lambda_j \varphi(c, ||x - x_j||),
\]

(11)

Where \( \lambda_j \) are the set of unknown coefficients which are usually determined by collocation with given discrete data, such as function values or derivative information. By choosing \( N \) collocation nodes \( x_j \) in the intervals \( (0, \infty) \) the function \( f^k \) can be interpolated as follow:
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\[ f^n(x_t, c) = \sum_{j=1}^{N} \lambda_j^2 \phi(c \| x_t - x_j \|), t = 1, \ldots, N \]  
(12)

The expansion coefficients \( \lambda_j^2 \) can therefore be obtained by solving the linear system \( \mathbf{A} \mathbf{V} = \mathbf{B} \), where \( \mathbf{A} = [\phi^T(x_1), \phi^T(x_2), \ldots, \phi^T(x_N)] \) and \( \mathbf{B} = [f^1(x_1, c), f^2(x_2, c), \ldots, f^N(x_N, c)] \).

And \( \mathbf{B} = [f^1(x_1, c), f^2(x_2, c), \ldots, f^N(x_N, c)] \).

Because of \( \phi_i(x_j) = \phi_i(x_j) \) consequently the matrix \( \mathbf{A} \mathbf{A}^T \) is symmetric. On the other hand according to investigations of (Schoenberg, IJ, 1938), (Sarra, SA, 2005) and (Micchelli. CA, 1986), \( \mathbf{A} \mathbf{A}^T \) is nonsingular i.e, it has an unique interpolates of the form Equation 2.11 and no matter how the distinct data points are scattered in any number of space dimensions.

**A mesh free method based on RBFs:**

For solving the Equation (8) with the boundary conditions (9) , the mesh free method based on the RBF is considered. It is supposed that the closed form approximating function Equation (11) obtained from a set of training points and its derivative of any order, that is:

\[ \frac{d^k f(\mathbf{\varphi})}{d\varphi^k} = \frac{d^k}{d\varphi^k} \left( \sum_{j=1}^{N} \lambda_j \phi_j(\mathbf{\varphi}) \right) = \sum_{j=1}^{N} \lambda_j \frac{d^k \phi_j(\mathbf{\varphi})}{d\varphi^k}, k = 1, \ldots, p \]  
(13)

Where \( p \) is the order of approximation derivative. By using Equation 13 in Equation with the boundary conditions 9, we obtain the following equations:

\[
\begin{aligned}
F(\mathbf{\varphi}, f, f', \ldots, f^{(p-1)}(\mathbf{\varphi})), \quad t = 0, \varphi \in (0, \varphi_0), \\
F^{(t)}(\mathbf{\varphi}_0) = \alpha_{t+1}, \quad t = 0, \ldots, p-1.
\end{aligned}
\]
(14)

Where \( \varphi_t \in (0, \varphi_0) \) and \( f(\mathbf{\varphi}_0) = \frac{d f(\mathbf{\varphi})}{d\varphi} \) and \( F \) is known function and \( \{x_j\}_{j=1}^{p} \) are known constants. By substituting Equation 11 in 14 and using 13 we have:

\[
F(\mathbf{\varphi}, f, f', \ldots, f^{(p-1)}(\mathbf{\varphi})) = F(\mathbf{\varphi}, \sum_{j=1}^{N} \lambda_j \phi_j(\mathbf{\varphi}), \sum_{j=1}^{N} \lambda_j \frac{d \phi_j(\mathbf{\varphi})}{d\varphi}, \ldots, \sum_{j=1}^{N} \lambda_j \frac{d^{p-1} \phi_j(\mathbf{\varphi})}{d\varphi^{p-1}}).
\]

To obtain the coefficient \( \mathbf{V} = [\lambda_1, \lambda_2, \ldots, \lambda_N]^T \), we define the residual function:

\[ R_{\mathbf{\varphi}}(\mathbf{\varphi}) = F(\mathbf{\varphi}, f, f', \ldots, f^{(p-1)}(\mathbf{\varphi})) \]

, then equalize Equation 15 to zero at N-p interpolate node with p boundary conditions, i.e:

\[
\begin{aligned}
R_{\mathbf{\varphi}}(\mathbf{\varphi}) &= 0, \quad j = 1, \ldots, N-p, \\
\sum_{j=1}^{N} \lambda_j \frac{d^p \phi_j(\mathbf{\varphi})}{d\varphi^p} &= \alpha_{t+1}, \quad k = 0,1, \ldots, p-1.
\end{aligned}
\]
(16)

Now we can approximation \( f(\varphi) \) by above explanation as:

\[ f(\varphi) = \sum_{j=1}^{N} \lambda_j \phi_j(\varphi). \]

Here \( f''(\varphi) \) and \( f''(\varphi) \) have defined as:

\[ f''(\varphi) = \frac{d^2 f(\varphi)}{d\varphi^2} = \sum_{j=1}^{N} \lambda_j \frac{d^2 \phi_j(\varphi)}{d\varphi^2}, \quad f''(\varphi) = \sum_{j=1}^{N} \lambda_j \frac{d^2 \phi_j(\varphi)}{d\varphi^2}. \]

To solve the problem:

\[
\begin{aligned}
R_{\mathbf{\varphi}}(\mathbf{\varphi}) &= F(\mathbf{\varphi}, f, f', \ldots, f^{(p)}(\mathbf{\varphi})), \\
f^{(0)}(\varphi) &= s, \\
f^{(\infty)}(\varphi) &= 0.
\end{aligned}
\]

By equalizing \( R_{\mathbf{\varphi}}(\mathbf{\varphi}) \) to zero at N-3 interpolate nodes \( \phi_i \in (0, \varphi_0) \).

**RESULT AND DISCUSSION**

In this section the presented technique is employed to approximate the similarity solutions of the Equation 8 with given boundary conditions 9 in both cases, \( m=1 \), the viscous flows over a two dimensional stretching surface, and \( m=2 \), the viscous flows over an axisymmetric stretching surface. The computed solutions of the similar stream function, \( f(\varphi) \), velocity distribution of flow, \( f'(\varphi) \) and the function of shear stress, \( f''(\varphi) \), for the case of the viscous flows over a two dimensional stretching surface, \( m=1 \) and by sitting \( k_p = 0.5, s = 2, \delta = -1, \gamma = 1 \) and for several values of \( M \) are obtained. To show the accuracy and applicability of our method, we compared the results of computed solutions with exact solutions in figures 1-3.
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Figure 1 Comparison between exact (dots) and approximate (line) solutions of the similar stream function, \( f(\phi) \) for case of the viscous flows over a two dimensional stretching surface, \( m=1 \) and by letting \( k_p = 0.5, s = 2, \delta = -1, \gamma = 1 \) and for several values of \( M \).

Figure 2 Comparison between exact (dots) and approximate (line) solutions of the velocity distribution of flow \( f'(\phi) \) for case of the viscous flows over a two dimensional stretching surface, \( m=1 \) and by letting \( k_p = 0.5, s = 2, \delta = -1, \gamma = 1 \) and for several values of \( M \).

Figure 3 Comparison between exact (dots) and approximate (line) solutions of the function of shear stress, \( f''(\phi) \) for case of the viscous flows over a two dimensional stretching surface, \( m=1 \) and by letting \( k_p = 0.5, s = 2, \delta = -1, \gamma = 1 \) and for several values of \( M \).

Figure 4 Comparison between exact (dots) and approximate (line) solutions of the similar stream function, \( f(\phi) \) for case of the viscous flows over a two dimensional stretching surface, \( m=1 \) and by letting \( M = 5, s = 2, \delta = -1, \gamma = 1 \) and for several values of \( k_p \).
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Figure 5 Comparison between exact (dots) and approximate (line) solutions of the velocity distribution of flow, $f(\phi)$ for case of the viscous flows over a two dimensional stretching surface, $m=1$ and by letting $M = 5$, $s = 2$, $\delta = -1$, $\gamma = 1$ and for several values of $k_p$.

Figure 6 Comparison between exact (dots) and approximate (line) solutions of the function of shear stress, $f'(\phi)$ for case of the viscous flows over a two dimensional stretching surface, $m=1$ and by letting $M = 5$, $s = 2$, $\delta = -1$, $\gamma = 1$ and for several values of $k_p$.

In Figures 4-6 the approximated results for $f(\phi)$, $f'(\phi)$ and $f''(\phi)$ for case of the viscous flows over a two dimensional stretching surface, $m=1$ and by choosing $M = 5$, $s = 2$, $\delta = -1$, $\gamma = 1$ and for various values of $k_p$ are plotted and compared with the exact solutions.

Figure 7. Comparison between exact (dots) and approximate (line) solutions of the velocity distribution of flow, $f''(\phi)$ for case of the viscous flows over a two dimensional stretching surface, $m=1$ and by letting $M = 5$, $k_p = 0.5$, $\delta = -1$, $\gamma = 1$ and for several values of $s$. 
Figure 8. Comparison between exact (dots) and approximate (line) solutions of the function of shear stress, $f''(\varphi)$ for case of the viscous flows over a two dimensional stretching surface, $m=1$ and by letting $M = 5$, $k_p = 0.5$, $\delta = -1$, $\gamma = 1$ and for several values of $s$.

Also in figures 7-8, obtained results for $f''(\varphi)$ and $f'''(\varphi)$ for $m=1$ and by choosing $M = 5$, $k_p = 0.5$, $\delta = -1$, $\gamma = 1$ and for several values of $s$ are plotted and compared with the exact solutions. From these presented results, it is clearly concluded that the approximated solutions computed from the RBF collocation method are in excellent agreement with the exact solutions.

Also the radial basis functions collocation method is applied for solving the governing problem for case of the viscous flows over an axisymmetric stretching surface, $m=2$. In this case, we clearly show the effect of different values of $m$, $k_p$, $s$, $\gamma$ and $\delta$ on $f'(\varphi)$, $f''(\varphi)$ and $f'''(\varphi)$. The numerical results are obtained for the region $\varphi \in (0, \varphi_w)$.

Figure 9. Profiles of the similar stream function, $f(\phi)$ for case of the viscous flows over an axisymmetric stretching surface, $m=2$ and by letting $k_p = 0.5$, $s = 2$, $\delta = -1$, $\gamma = 1$ and for several values of $M$.

Figure 10. Profiles of the velocity distribution of flow $F(\varphi)$ for case of the viscous flows over an axisymmetric stretching surface, $m=2$ and by letting $k_p = 0.5$, $s = 2$, $\delta = -1$, $\gamma = 1$ and for several values of $M$. 
Figure 11. Profiles of the variation of the shear stress, \( f''(\phi) \) for case of the viscous flows over an axisymmetric stretching surface, \( m=2 \) and by letting \( k_p = 0.5, s = 2, \delta = -1, \gamma = 1 \) and for several values of \( M \).

Figure 12. Profiles of the similar stream function \( f(\phi) \) (left), velocity distributions \( f'(\phi) \) (center) and variation of the shear stress, \( f''(\phi) \), (right) for case of the viscous flows over an axisymmetric stretching surface, \( m=2 \) and by letting \( M = 5, s = 2, \delta = -1, \gamma = 1 \) and for several values of \( k_p \). Figures 9-11 depict effect of \( M \) on \( f'(\phi), f''(\phi) \) and \( f'''(\phi) \). From Figure 9 can be understood that by decreasing value of investigated parameter \( M \), \( f'(\phi) \) increases, as it is shown in Figure 11. Whenever \( M \) decreases, \( f''(\phi) \) decreases too. These results are similar to Figures 1-3, when \( m=1 \). According to Figure 12 whenever \( k_p \) increases, \( f'(\phi) \) increases too, but \( f''(\phi) \) decreases. The thickness layer decreases with decreasing \( k_p \).

Figure 13. Profiles of the velocity distributions \( f(\phi) \) (left) and variation of the shear stress \( f''(\phi) \) (right) for case of the viscous flows over an axisymmetric stretching surface, \( m=2 \) and by letting \( M = 5, k_p = 0.5, \delta = -1, \gamma = 1 \) and for several values of \( s \).

Figure 14. Profiles of the similar stream function \( f(\phi) \) (left), velocity distributions \( f'(\phi) \) (center) and variation of the shear stress, \( f''(\phi) \) (right) for case of the viscous flows over an axisymmetric stretching surface, \( m=2 \) and by letting \( M = 5, s = 2, \delta = -1, k_p = 1 \) and for several values of \( \gamma \).
Figure 15. Profiles of the similar stream function \( f(\varphi) \) (left), velocity distributions \( f(\varphi) \) (center) and variation of the shear stress, \( f'(\varphi) \) (right) for case of the viscous flows over an axisymmetric stretching surface, \( m=2 \) and by letting \( M=5, \gamma=2, k_{p}=1 \) and for several values of \( \delta \).

For all plots, it can be understood that \( f'(\varphi) \), as a function of \( \varphi \), is negative for all values of all parameters, \( M, s, \gamma, k_{p} \), and \( \delta \). Comparing Figures 7-8 and 13 can be deduced that decreasing \( |s| \), increasing \( f'(\varphi) \) while \( s \) has a direct effect on \( f'(\varphi) \), i.e., fluid flow at nano scales and \( f'(\varphi) \) at the wall decrease with a decreasing in \( |s| \). According to profile of Figure 14, effect on \( f(\varphi) \), \( f(\varphi) \) inversely while \( f'(\varphi) \) decreasing rapidly. In Figure 15 the effect of \( \delta \) on the \( f(\varphi) \), \( f'(\varphi) \) and is shown when \( M=5, s=2, \gamma=1, k_{p}=1 \) when \( m=2 \). When \( \delta \) decreases the thickness layout and velocity distribution of flow increased considerably, while \( f''(\varphi) \) decreases, same as before.

CONCLUSION
In this study the IMQ RBF collocation method is employed for the solution of boundary-layer problems is presented. The numerical solutions are given for different values of model's parameters. The effects of parameters on the Nano boundary thickness is investigated and at the end treatment of similar stream function, velocity distribution and shear stress are plotted by variable parameters \( k_{p}, \gamma, \delta \) and \( M \). The figures show that the computed solutions are in good agreements with the exact solutions.

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