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**Research Article**

# Iterative Reconstruction in Continuous Frame Theory

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**ABSTRACT.**

Sampling theorem combined with an efficient reconstruction algorithm, has been applied successfully to proveirregular sampling theorems in spaces of analytic functions [12], for shorttime Fourier transforms and wavelet transforms [13], and for a general classof spaces of band-limited functions [14,15]. An important instance of this algorithm occurs in the presence of framesin a Hilbert space. We present a newmethod to obtain continuous framebounds. In the applicationsto irregular sampling of band-limitedfunctions this alternative strategyleads to better and explicit estimatesof the continuous frame bounds.

**Keywords:** Operator Bounded Operator, Iterative Reconstruction, Frame, Frame Operator.

## 1. INTRODUCTION

Frames was first introduced by Duffine and schaeffer[9] in the context of nonharmonic Fourier series. Outsideof signal processing, frames did not seem to generate much interest until the ground breaking work of Daubechies, Grossman, and Meyer [8] in 1986. Since then the theory of frames began to be more widely studied. During the last 20 years the theory of frames has been growing rapidly, since several new applications have been developed. For example, besides traditional application as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in pocket-based communication systems and hence to improve the robustness of data transmission, and to design high-rate constellation with full diversity in multiple-antenna code design.

The organization of this article is as follows. In section 2, the generic form of many reconstruction methods, the definition of frame and its fundamental properties in discreet and continuous version will be given.In section 3, with name main result,we present a newmethod to obtain continuous framebounds. In the applicationsto irregular sampling of band-limitedfunctions this alternative strategyleads to better and explicit estimatesof the continuous frame bounds.

## 2. PRELIMINARIES

In this section we discuss the generic form of many reconstruction methodsthat are currentlyused in signal analysis as well as in other areas of mathematics.This class of algorithms isdistinguished (a) by their linearity, (b)by their being iterative, and (c) by the geometricconvergence of successive approximations. From the functional analytic point of view theyare just theirversion of a linear operator by a Neumann series.A particular case of thesemethods are the algorithms based on frames.In [16], a new method is obtained for constructing frames which will be very useful obtain explicit estimates.In thisarticle we extended it to continuous version.First we review the following proposition from [18] with its proof.

**Proposition 2.1.**Let $A$ be a bounded operator on a Banach space $(B, \|.\|_B)$ that satisfies for some positive constant $\gamma < 1$

$$\|f - Af\|_B \leq \gamma \|f\| \text{ for all } f \in B. \qquad (1)$$

Then $A$ is invertible on $B$ and $f$ can be recovered from $Af$ by the following iteration algorithm. Setting $f_0 = Af$ and

$$f_{n+1} = f_n + A(f - f_n) \qquad (2)$$

for $n \geq 0,$ we have

$$\lim_{n\to\infty} f_n = f \qquad (3)$$

With the error estimate after $n$ iterations

$$\|f - f_n\|_B \leq \gamma^{n+1} \|f\|_B. \qquad (4)$$

Proof. By inequality (1) the operator norm of $Id - A$ is less than $\gamma$.This implies that $A$ is invertible and that the inverse can be presented as a Neumann series:

$$A^{-1} = \sum_{n=0}^{\infty} (Id - A)^n$$

and any $f \in B$ is determined by $Af$ and the norm-convergent series

$$f = A^{-1}Af = \sum_{n=0}^{\infty} (Id - A)^n Af.$$

The reconstruction (3) and the error estimate (4) follow easily after we have shown that the $n-th$ approximation $f_n$ as defined in (2) coincides with the $n-th$ partial sum $\sum_{k=0}^{n} (Id - A)^k Af$. This is clear for $n = 0,$ since $f_0 = Af$ by definition. Next assume that we know already that

$$f = \sum_{k=0}^{n} (Id - A)^k Af.$$

Then we obtain for $n+1$

$$\sum_{k=0}^{n+1} (Id - A)^k Af = Af + \sum_{k=1}^{n+1} (Id - A)^k Af$$

$$= Af + (Id - A)\sum_{k=0}^{n} (Id - A)^k Af \qquad (by \quad induction)$$

$$= Af + (Id - A) f_n = f_n + A(f - f_n).$$

Now clearly $\lim_{n\to\infty} f_n = f$ and since $\sum_{k=n+1}^{\infty} (Id - A)^k = (Id - A)^{n+1} A^{-1},$ we obtain

$$\|f - f_n\|_B = \left\| \sum_{k=n+1}^{\infty} (Id - A)^k Af \right\|_B = \|(Id - A)^{n+1} A^{-1} Af\| \leq \gamma^{n+1} \|f\|_B.$$

In thefollowing we review the definitions and some important properties of frame theory in discrete and continuous frames. We will see that the continuous frame is an interest extension of frame which provided new tools for study the exquisite operators on a complex Hilbert space.

**Definition 2.2.** Let $\{f_i\}_{i\in I}$ be sequence of members of $H$. We say that $\{f_i\}_{i\in I}$ is a frame for $H$ if there exist $0<A\le B<\infty$ such that for all $h\in H$,

$$A\|h\|^2\le\sum_{i\in I}|<f_i,h>|^2\le B\|h\|^2.$$

The constants $A$ and $B$ are called frame bounds. If $A,B$ can be chosen so that $A=B$, we call this frame an $A-tight$ frame and if $A=B=1$ it is called a parseval frame. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a Bessel sequence. If $\{f_i\}_{i\in I}$ is a Bessel sequence then the following operators are bounded and $l^2(I)$ is called representation space in discrete version of frame theory.

$T:l^2(I)\to H,\ T(c_i)=\sum_{i\in I}c_if_i$ (synthesis operator),

$T^*:H\to l^2(I),\ T^*f=\{<f,f_i>\}_{i\in I}$ (analysis operator),

$S:H\to H,\ Sf=TT^*f=\sum_{i\in I}<f,f_i>f_i$ (frame operator).

**Definition 2.3.**Let $(X,\mu)$ be a measure space and $\mu$ be a positive measure. Let $F:X\to H$ be weakly measurable (i.e., for all $h\in H$, the mapping $x\to<F(x),h>$ is measurable). Then $F$ is called a continuous frame for $H$ if there exist $0<A\le B<\infty$ such that, for all $h\in H$,

$$A\|h\|^2\le\int_X|<h,F(x)>|^2d\mu\le B\|h\|^2 \qquad (5).$$

The constants $A$ and $B$ are called continuous frame bounds. If $A,B$ can be chosen so that $A=B$, we call this frame an $A-tight$ continuous frame and if $A=B=1$ it is called a parsevalcontinuous frame. If we only have the upper bound, we call $F$ a Bessel mapping. The representation space in continuous frame is chosen $L^2(X,\mu)$ and If $F$ is Bessel mapping the following operators are bounded.

$T:L^2(X,\mu)\to H,\ \langle T(\varphi),h\rangle=\int_X\varphi(x)<F(x),h>d\mu$ (synthesis operator),

$T^*:H\to L^2(X,\mu),\ (T^*f)(x)=\langle h,F(x)\rangle$ (analysis operator),

$S_F:H\to H,\ Sf=TT^*f$ (frame operator).

By definition of frame operator, for each $h\in H$ we have

$$\langle S_F(h),h\rangle=\int_X|<h,F(x)>|^2d\mu. \qquad (6)$$

Since $AI\le S_F\le BI$, then $S_F$ is a positive, invertible and self adjoint operator on $H$.

## 3. MAIN RESULT

**Definition 3.1.** Let $F:X\to H$ be a continuous frame with bounds $A,B$ and frame operator $S_F$.We define quasi frame operator $\Gamma_F$ for continuous frame $F$ as follows

$$\Gamma_F(h)=\frac{2}{A+B}\int_X<h,F(x)>F(x)d\mu.$$

It is clear that $\Gamma_F$ is positive. Now, let $Id$ be the identity operator on $H$ and we consider the self-adjoint operator $Id-\Gamma_F$ on $H$.

**Lemma 3.2.** Let $\Gamma_F$ be the quasi frame operator for continuous frame $F$, then the operator norm of $Id-\Gamma_F$ is smaller than 1 and we have

$$-\frac{B-A}{B+A}\|h\|^2 \le \langle (Id-\Gamma_F)(h),h\rangle \le \frac{B-A}{B+A}\|h\|^2.$$

Proof. By (5) we have

$$-\frac{B-A}{B+A}\|h\|^2 = \|h\|^2 - \frac{2B}{B+A}\|h\|^2 \qquad (7)$$

$$\le \|h\|^2 - \frac{2}{B+A}\int_X |<h,F(x)>|^2 d\mu$$

$$= \|h\|^2 - \langle \Gamma_F(h),h\rangle$$

$$= \langle (Id-\Gamma_F)(h),h\rangle$$

$$\le \|h\|^2 - \frac{2A}{B+A}\|h\|^2$$

$$\le \frac{B-A}{B+A}\|h\|^2.$$

Since $Id-\Gamma_F$ is self-adjoint, (7) entails the operator norm of $Id-\Gamma_F$ on $H$ is smaller than $\frac{B-A}{B+A}<1.$

**Remark 3.3.** Now proposition 2.1 applies, and yields an iterative reconstruction of $h$ by $\Gamma_F(h)$ with the rate of convergence determined by $\gamma=\frac{B-A}{B+A}$. Since the input for $\Gamma_F(h)$ consists of the frame coefficients $<h,F(x)>$ only this is indeed the desired reconstruction. After setting $G=S_F^{-1}$, we may write the reconstructions of $h$ as follow

$$h=S_F^{-1}S_F(h)=\int_X <h,F(x)>S_F^{-1}(F(x))d\mu,$$

$$h=S_FS_F^{-1}(h)=\int_X <h,S_F^{-1}(F(x))>F(x)d\mu.$$

Note that the two frame bounds $A$ and $B$ play a vital role in the algorithm and determine the speed of convergence of the algorithm. It is therefore of practical importance to estimate $A$ and $B$ as sharp as possible. One is content with the mere existence of frame bounds $A$ and $B$ considers the family the family of operators that we define as follows.

**Definition 3.4.**Let $F:X\to H$ be a continuous frame with bounds $A,B$ and frame operator $S_F$. We define $\lambda-$quasi frame operator $\Gamma_{\lambda F}$ for continuous frame $F$ as follows

$$\Gamma_{\lambda F}(h)=\lambda\int_X <h,F(x)>F(x)d\mu,$$

where $\lambda$ is the so-called relaxation parameter.

**Remark 3.5.**With an estimate similar to Lemma 3.2 one obtains

$$\|h-\Gamma_{\lambda F}\| \le \gamma(\lambda)\|h\|,$$

where $\gamma(\lambda)=\max\{1-\lambda A,\quad 1-\lambda B\}$ and $\gamma(\lambda)<1$ for small values of $\lambda$. Note that the above inequality obtained by similar method which were told in Lemma 3.2.

Here we present a new method to obtain frame bounds. In application to irregular sampling of band-limited functions this alternative strategy leads to better and explicit estimates of the frame bounds in continuous version.

**Definition 3.6.**Let $(X, \mu)$ be a measure space and $\mu$ be a positive measure. Let $F: X \to H$ and $G: X \to H$ are weakly measurable (i.e., for all $h \in H$, the mapping $x \to < F(x), h >$ and $x \to < G(x), h >$ are measurable). We define approximation operator $A_{FG}: H \to H$ with respect to $F$ and $G$ as follows

$$A_{FG}(h) = \int_X < h, F(x) > G(x) d\mu.$$

**Theorem 3.7.**Suppose that $F$ and $G$ be defined as in Definition 3.6, for each $h \in H$ and $\varphi \in L^2(X, \mu)$ we have

(i) $\int_X |< h, F(x) >|^2 d\mu \le C_1 \| h \|^2$ (8),

(ii) $\left\| \int_X \varphi(x) G(x) d\mu \right\|^2 \le C_2 \| \varphi \|_2^2$ (9),

(iii) $\left\| h - \int_X < h, F(x) > G(x) d\mu \right\| \le \gamma \| h \|.$ (10)

Then $F$ is a continuous frame with bounds $(1-\gamma)^2 / C_2$ and $C_1$, also $G$ is a continuous frame with bounds $(1-\gamma)^2 / C_1$ and $C_2$.

Proof. Let $A_{FG}$ be defined as in Definition 3.6, then $A_{FG}$ is bounded operator on $H$ because for each $h \in H$, assuming $\varphi(x) = < h, F(x) >$ then (8) results in $\varphi \in L^2(X, \mu)$, and by (8) and (9) we have

$$\left\| A_{FG}(h) \right\|^2 = \left\| \int_X < h, F(x) > G(x) d\mu \right\|^2 \le C_2 \int_X |< h, F(x) >|^2 d\mu \le C_1 C_2 \| h \|^2 .$$

By proposition 2.1 $A_{FG}$ is invertible with $A_{FG}^{-1} = \sum_{n=0}^{\infty} (Id - A_{FG})^n$ and $\left\| A_{FG}^{-1} \right\| \le (1-\gamma)^{-1}$.

Now by (8) and (9) we have

$$\|h\|^2 = \left\| A_{FG}^{-1} A_{FG}(h) \right\|^2 \le (1-\gamma)^{-2} \left\| A_{FG}(h) \right\|^2$$

$$= (1-\gamma)^{-2} \left\| \int_X < h, F(x) > G(x) d\mu \right\|$$

$$\le C_2 (1-\gamma)^{-2} \int_X |< h, F(x) >|^2 d\mu \le C_1 C_2 (1-\gamma)^{-2} \| h \|^2 .$$

We conclude that $F$ has required properties.

Next we verify two inequalities which are dual to (8) and (9),

$$\left(\int_X |< h, G(x) >|^2 d\mu\right)^2 = \left(\left\langle \int_X < h, G(x) > G(x) d\mu, h \right\rangle\right)^2$$

$$\le \left\| \int_X < h, G(x) > G(x) d\mu \right\|^2 \| h \|^2 \le C_2 \| h \|^2 \int_X \left|< h, G(x) >\right|^2 d\mu,$$

hence

$$\int_X |< h, G(x) >|^2 d\mu \le C_2 \| h \|^2 .$$

For second inequality we have

$$\left\|\int_X \varphi(x)F(x)d\mu\right\| = \sup_{\|h\|=1}\left|\left\langle h, \int_X \varphi(x)F(x)d\mu\right\rangle\right|,$$

and

$$\left|\left\langle h, \int_X \varphi(x)F(x)d\mu\right\rangle\right|^2 = \left|\int_X \varphi(x)\left\langle h, F(x)\right\rangle d\mu\right|^2 \le C_1 \|\varphi\|_2^2 \|h\|^2.$$

Now by similar argument and applying an approximation operator of the form

$$A_{GF}(h) = \lambda \int_X < h, G(x) > F(x)d\mu,$$

we can establish $G$ has required properties.

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