International Journal of Advanced Computer and Mathematical Sciences
ISSN 2230-9624. Vol 3, Issue 2, 2012, pp 237-244
http://bipublication.com

A NUMERICAL SOLUTION OF THE FLOW OF TWO IMMISCIBLE FLUIDS THROUGH POROUS MEDIA WITH MEAN PRESSURE

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[Received-30/03/2011, Accepted- 07/05/2012]

ABSTRACT

The present paper discusses the phenomenon of flow of two immiscible liquid through homogenous porous media with mean capillary pressure. The underlying basic assumption made in the present analysis is that the individual pressure of the two flowing phases may be replaced by their common mean pressure and some standard relation has been used. The mathematical formulation leads to Boussinesq's Equation which has been solved by using a numerical technique by employing the iterative process with help of appropriate boundary conditions. This problem has great importance in petroleum technology.

Key words: Immiscible, Homogenous porous media, Capillary pressure, Boussinesq's Equation

INTRODUCTION

The oil-water movement in a porous medium is an important problem of petroleum technology and water hydrology (Scheidegger 1966). Here we consider that the injection of water into an oil formation in porous medium providing a two phase liquid-liquid flow problem. Such a problem is generally encountered in secondary recovery process.

Many researchers have discussed this phenomenon with different points of view. Some of these are summarised here. Scheidegger (1960) considered the average cross-sectional area occupied by the fingers while the size and shape of the individual fingers were neglected. It was shown by Scheidegger and Johnson (1961) that treatment of motion with concept of fictitious relative permeability is formally identical to the Buckley Leverett(1942) description of two immiscible fluids flowing through the porous medium. Most of the earlier researcher such as Saffman and Taylor (1958), Scheidegger and Johnson

(1961), Wodding(1962) have completely neglected the capillary pressure. Verma(1970) included capillary pressure in the analysis of fingers. Verma (1969) has discussed the statistical behaviour of the fingering phenomenon in a displacement process in heterogeneous porous medium with capillary pressure using perturbation solution.

In this paper we assume that the individual pressure of two flowing phases may be replaced by their common mean pressure (Oroveanu 1964) and the expression for the phase saturation distribution are obtained. The mathematical formulation leads to Boussinesq's Equation which has been solved numerically.

Our particular interest in the present paper is to obtain numerical solution for phase saturation by using an iterative process. The numerical solution of the nonlinear differential equation of the two immiscible liquids flow has been obtained.

MATHEMATICAL FORMULATION

The seepage velocity of water(V_w) and oil(V_o) may be written as (by Darcy's law)

$$V_w = -\frac{K_w}{r_w} K \frac{\partial P_w}{\partial x} \quad (1)$$

$$V_o = -\frac{K_0}{r_o} K \frac{\partial P_0}{\partial x} \quad (2)$$

where K is the permeability of the homogeneous medium, K_w and K_0 are the relative permeability of water and oil, which are functions of S_w and S_0 (S_w and S_0 are the saturations of water and oil respectively), p_w and p_0 denote the pressure of water and oil, while r_w and r_0 are the constant kinematic viscosities of the phases.

The equations of continuity (phase densities are constant) are

$$m\frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \quad (3)$$

$$m\frac{\partial S_0}{\partial t} + \frac{\partial V_0}{\partial x} = 0 \quad (4)$$

where m is the porosity of medium. From the definition of phase saturation, it is evident that

$$S_w + S_o = 1 \quad (5)$$

The capillary pressure (P_c), defined as the pressure discontinuity of the flowing phases across their common interface may be written as

$$P_c = P_0 - P_w \quad (6)$$

The equation of motion for saturation can be obtained by substituting the values of V_w and V_0 from equations, (1) and (2) in eqns. (3) and (4) respectively, we get,

$$m\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{K_w}{r_w} K \frac{\partial P_w}{\partial x}\right) \quad (7)$$

$$m\frac{\partial S_0}{\partial t} = \frac{\partial}{\partial x}\left(\frac{K_o}{r_o} K \frac{\partial P_o}{\partial x}\right) \quad (8)$$

Eliminating $\frac{\partial p_w}{\partial x}$ from eqns, (6) and (7), we obtain

$$m\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left[\frac{K_w}{r_w} K \left(\frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x}\right)\right] \quad (9)$$

Combining equations (2.8) and (2.9) and using eqn. (2.5), we have

$$\frac{\partial}{\partial x}\left[\left(\frac{K_w}{r_w}K+\frac{K_o}{r_o}K\right)\frac{\partial p_o}{\partial x}-\frac{K_w}{r_w}K\frac{\partial p_c}{\partial x}\right]=0 \quad (10)$$

Integrating eqn. (10) with respect to *x*, we get

$$\left(\frac{K_w}{r_w}K+\frac{K_o}{r_o}K\right)\frac{\partial p_o}{\partial x}-\frac{K_w}{r_w}K\frac{\partial p_c}{\partial x}=-V \quad (11)$$

where *V* is a constant of integration which can be evaluated later on.

By simplifying (11), we get

$$\frac{\partial P_0}{\partial x}=\frac{1}{1+\frac{K_0}{K_w}\cdot\frac{r_w}{r_0}}\frac{\partial P_c}{\partial x}-\frac{V}{\left(\frac{K_w}{r_w}K+\frac{K_o}{r_o}K\right)} \quad (12)$$

Putting the value of $\frac{\partial p_o}{\partial x}$ in equation (9), we have

$$m\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial x}\left|\frac{K_0/r_o}{1+\frac{K_0}{K_w}\cdot\frac{r_w}{r_0}}\frac{\partial P_c}{\partial x}+\frac{V}{1+\frac{K_0}{K_w}\cdot\frac{r_w}{r_0}}\right|=0 \quad (13)$$

The value of pressure of oil (p_0) can be written as (Oroveanu 1966)

$$p_o=\frac{p_n+p_w}{2}+\frac{p_n-p_w}{2}=\bar{p}+\frac{1}{2}p_c \quad (14)$$

where $\bar{p}$ is the mean pressure which is constant and

$$\frac{\partial p_o}{\partial x}=\frac{1}{2}\frac{\partial p_c}{\partial x} \quad (15)$$

Substituting this value of $\frac{\partial p_o}{\partial x}$ in eqn. (11), we get

$$V=\frac{1}{2}\left[\left(\frac{K_w}{r_w}K-\frac{K_o}{r_o}K\right)\right]\frac{\partial p_c}{\partial x} \quad (16)$$

Then by substituting the value if *V* from eqn. (16) into (13) we obtain

$$m\frac{\partial S_w}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{K_w}{r_w}K\frac{dp_c}{dS_w}\frac{\partial S_w}{\partial x}\right)=0. \quad (17)$$

At this stage definiteness of the mathematical analysis we assume a standard form of Jones (1961) for the relationship between the relative permeability phase saturation and a linear relationship between capillary pressure phase saturation as

$$K_w=S_w^3,\ K_0=S_0=1-\alpha_0 S_w,\ (\alpha_0=1.11) \quad (18)$$

$$p_c=\beta\,(S_w^{-1}-C),\ (\beta \text{ and } C \text{ are constants}). \quad (19)$$

Substituting this value in eqn. (17), we get

$$m\frac{\partial S_w}{\partial t}-\frac{1}{2}\frac{\beta}{r_w}K\frac{\partial}{\partial x}\left(S_w\frac{\partial S_w}{\partial x}\right)=0 \quad (20)$$

Equation (20) can be converted into a dimensionless form by considering a variable

$$X = \frac{x}{L}, T = \frac{3}{2}\frac{\beta K t}{m L^2 r_w} \tag{21}$$

Then equation (20) takes the form

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left(S_w \frac{\partial S_w}{\partial x}\right) \tag{22}$$

This is the nonlinear differential equation of motion for saturation subject to the conditions

$$S_w(0,T) = S_0 , \quad T > 0 \tag{23}$$

$$S_w(X,0) = S_n , \quad X > 0 \tag{24}$$

NUMERICAL SOLUTION

The transformation

$$\Phi = \frac{X}{\sqrt{T}} \tag{25}$$

allows (22) to be reduced to the ordinary differential equation

$$-\frac{\phi}{2}\frac{dS_w}{d\phi} = \frac{d}{d\phi}\left(S_w \frac{dS_w}{d\phi}\right) \tag{26}$$

Multiplying both sides of (26) by $d\phi / d\theta$ gives

$$-\frac{\phi}{2} = \frac{d}{dS_w}\left(S_w \frac{dS_w}{d\phi}\right) \tag{27}$$

subject to the conditions

$$S_w = S_0, \phi = 0 (for\ x=0);\ S_w \to S_n \quad as\ S_w \to 1 \tag{28}$$

which imply

$$S_w \to S_n ,\ dS_w/d\phi \to 0, \tag{29}$$

A suitable form of (27) for subsequent treatment is obtained by integrating

$$\int_{S_n}^{S_w} \phi\, dS_w = 2 S_w \frac{dS_w}{d\phi} \tag{30}$$

The lower terminal of the integral is fixed as S_n by (29).

(30) is subject to the condition, $S_w = S_0 \quad for \quad \phi = 0\ (i.e.\ x=0)$ (31)

FINITE DIFFERENCE FORM

Let the interval S_o to S_n be divided into n equal steps δS_w. Let

$$S_r = S_0 + r\delta S_w, \tag{32}$$

and let the suffix r append to a quantity donate its value $S_w = S_r$.

Replacing the $\phi(S_w)$ curve by the histogram with (ϕ_r, S_r) as co-ordinates of the midpoint of step r enables the development of the simple finite difference approximations:

$$\int_{S_r}^{S_{r+\frac{1}{2}}} \phi \, dS_w = \frac{\phi_r \delta S_w}{2} \qquad (33)$$

$$\int_{S_{r-\frac{1}{2}}}^{S_r} \phi \, dS_w = \frac{\phi_r \delta S_w}{2} \qquad (34)$$

Additionally, we shall employ the approximation

$$\left(S_w \frac{dS_w}{d\phi}\right)_{r+\frac{1}{2}} = \frac{\overline{S_w}_{(r+\frac{1}{2})} \ \delta S_w}{\phi_{r+1} - \phi_r} \qquad (35)$$

where

$$\overline{S_w}_{(r+\frac{1}{2})} = \frac{\int_{S_r}^{S_{r+1}} S_w \, dS_w}{\int_{S_r}^{S_{r+1}} dS_w} = \frac{1}{2}\,(S_{r+1} + S_r)$$

Therefore,

$$\left(S_w \frac{dS_w}{d\phi}\right)_{r+\frac{1}{2}} = \frac{(S_{r+1} + S_r) \ \delta S_w}{2(\phi_{r+1} - \phi_r)} \qquad (36)$$

Substituting (33) and (36) in (30) for S_w = $S_{r+1/2}$, the result is

$$\int_{S_n}^{S_r} \phi \, dS_w + \frac{\phi_r \delta S_w}{2} = \frac{(S_{r+1} + S_r) \ \delta S_w}{(\phi_{r+1} - \phi_r)} \qquad (37)$$

Equation (37) may be rewritten

$$\phi_{r+1} - \phi_r = \frac{(S_{r+1} + S_r)}{I_{r+\frac{1}{2}}} \qquad (38)$$

where

$$I_{r+\frac{1}{2}} = \frac{1}{\delta S_w} \int_{S_n}^{S_r} \phi \, dS_w + \frac{\phi_r}{2} \qquad (39)$$

Similarly,

$$I_{r-\frac{1}{2}} = \frac{1}{\delta S_w} \int_{S_n}^{S_r} \phi \, dS_w - \frac{\phi_r}{2} \qquad (40)$$

From (39) and (40) , we get

$$I_{r+\frac{1}{2}} - I_{r-\frac{1}{2}} = \phi_r \qquad (41)$$

CONVERGENCE OF THE METHOD:

Consider some range $S_w = S_{n-\varepsilon}$ to $S_w = S_n$ where ε is such that the variation of S_w in the range is sufficiently small for the assumption S_w = constant = $\frac{1}{\varepsilon \delta S_w} \int_{S_{n-\varepsilon}}^{S_n} S_w \, dS_w$ to result in negligibly small errors. Usually, it will be practice and convenient to adopt $\varepsilon = 1$, the value taken in the following discussion. Evaluation of ϕ in the range S_{n-1} to S_n then reduces to solving

$$\frac{\partial S_w}{\partial t} = \overline{S_{w}}_{n-\frac{1}{2}} \frac{\partial^2 S_w}{\partial x^2}, \qquad (42)$$

Subject to the conditions

$$S_w = S_{n-1},\ x = \phi_{n-1} t^{\frac{1}{2}},\ S_w \to S_n,\ \phi \to \infty \qquad (43)$$

The analytical solution (Philip 1955) is,

$$\phi = 2(S_{w_n\ \frac{1}{2}})^{\frac{1}{2}}\ inverfc \left[\frac{S_w - S_n}{\delta S_w}\ erfc\ [\phi_{n-1}/2(S_{w_n\ \frac{1}{2}})^{1/2}\right] \qquad (44)$$

where the notation inverfc is proposed for the inverse of the function

$$erfc\ x = \frac{2}{\pi^{\frac{1}{2}}} \int_x^{\infty} \exp(\ \ \xi^2)\, d\xi. \qquad (45)$$

(44) is the equation to be used to evaluate ϕ close to $S_w = S_n$.

Integrating (44) and using (39) we obtain

$$I_{n-\frac{1}{2}} = \frac{\phi_{n\ \ 1}}{2} + 2(\overline{S_w}_{n-\frac{1}{2}})^{\frac{1}{2}} \frac{\text{ierfc}\left[\phi_{n-1}/2\left(\overline{S_w}_{n-\frac{1}{2}}\right)^{1/2}\right]}{\text{erfc}\left[\phi_{n-1}/2\left(\overline{S_w}_{n-\frac{1}{2}}\right)^{1/2}\right]} \qquad (46)$$

where, $ierfc\, x = \int_x^{\infty} erfc\, \xi\, d\xi = \frac{1}{\pi^{\frac{1}{2}}} exp(-x^2) - x\ erfc\ x \qquad (47)$

For evaluation of the second term of (46), it is convenient to investigate the function *A*, defined by

$A(x) = 2x\ ierfc\ x/erfc\ x. \qquad (48)$

The function 2 ierfc *x/erfc* x is in some ways more useful, but does not lend itself as readily to interpolation and accurate graphical representation over the desired range.

(48) tends to an alternative form of *A*

$$A(x) = \frac{2x\ exp(-x^2)}{\pi^{\frac{1}{2}} erfc\ x} - 2x^2 \qquad (49)$$

Employing the asymptotic expansion (large *x*) of erfc in (49) and simplifying, we obtain the asymptotic expansion (large *x*) of *A(x)*:

$$A(x) = 1 - \frac{2}{(2x^2)} + \frac{10}{(2x^2)^2} - \frac{74}{(2x^2)^3} + \frac{706}{(2x^2)^4} - \frac{8162}{(2x^2)^5} + \frac{108830}{(2x^2)^6} - \cdots. \qquad (50)$$

Values of *A(x)* for *x=0(0.2)1(0.5)4(1.0)10* are presented in table1. Values of *A* for $0 \leq x < 3$ were computed using (49) and a table of erf *x*. For *x>3*, insufficient significant figures in [1-erf x] (i.e. erfc x) were available from the table and the asymptotic expansion (50) was employed. In all computations sufficient decimals were retained to establish the third decimal place in *A* with certainty.

Table -1 The function *A*(*x*) = 2*x*ierfc *x/erfc x*

x	*A*	*x*	*A*	*x*	*A*
0.0	0.000	1.5	0.763	5.0	0.964
0.2	0.199	2.0	0.836	6.0	0.974
0.4	0.355	2.5	0.881	7.0	0.981
0.6	0.472	3.0	0.911	8.0	0.985
0.8	0.566	3.5	0.930	9.0	0.988
1.0	0.639	4.0	0.946	10.0	0.990

Two additional properties of *A* are:

x=0, dA/dx = 2 ierfc 0 = $2\pi^{-\frac{1}{2}}$ = 1.1284 (51)

and

$$\lim_{x\to\infty} A = 1 \tag{52}$$

Reverting to (46), we see that this may be rewritten

$$I_{n-\frac{1}{2}} = \frac{\phi_{n-1}}{2} + \frac{2\overline{S_{w}}_{n-\frac{1}{2}}}{\phi_{n-1}} A\left[\frac{\phi_{n-1}}{2(\overline{S_{w}}_{n-\frac{1}{2}})^{\frac{1}{2}}}\right] \tag{53}$$

1. THE ITERATIVE PROCEDURE

The iterative procedure can now be outlined, operations being listed in their order of performance:

(i) Tabulate $\overline{S_{w}}_{(r+\frac{1}{2})}$ from the known values of S_w.

(ii) Adopting a trial

$$I_{\frac{1}{2}}\left(=\frac{1}{\delta S_w}\int_{S_n}^{S_0}\phi\, dS_w\right)$$

use the condition $\phi_0 = 0$ and (38) to compute ϕ_1.

(iii) Use $I_{\frac{1}{2}}$ and ϕ in (41) to evaluate $I_{\frac{3}{2}}$

(iv) Continue the alternate use of (38) and (41) to evaluate ϕ_r and $I_{r+1/2}$ for integral values of r from 1 to n-1.

(v) Compare the value of $I_{n-1/2}$ given by this procedure $I_{n-\frac{1}{2}}{}^{*}$ with that given by (53),

$I_{n-1/2}{}^{\dagger}$.

(vi) Repeat (ii) to (v), using improved estimates of $I_{\frac{1}{2}}$ until $I_{n-\frac{1}{2}}{}^{*} - I_{n-\frac{1}{2}}{}^{\dagger}$ is sufficiently small.

CONCLUSION

The governing equation of flow of two immiscible fluids through Porous Media with mean pressure is equation (22) which is in dimensionless variable with approximate boundary condition and using standard relation have been solved by using appropriate transformation which has been converted into finite difference scheme (41) and its convergence method is employed and its analytical solution is given by equation (44) in terms of complimentary error function. The term A(x) is included in solution which helps the saturation of injected liquid by asymptotic expansion (50). It is observed that when x is very large, $A(x) \to 1$ which means that the saturation of injected liquid will approach to 1 which is given by table-1 and which is consistent with the physical phenomenon. The numerical work can be done by using iterative procedure which is given in paper. Here it is assumed that diffusivity coefficient is constant.

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