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# TRANSIENT ANALYSIS OF TANDEM QUEUEING MODEL WITH NON-HOMOGENOUS POISSON BULK ARRIVALS HAVING STATE-DEPENDENT SERVICE RATES

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## ABSTRACT
Queueing models provides the basic frame work for analysing the practical situations in production processes, communication networks, transportation systems, and machine repairing. In many of the communication systems the arrivals are time-dependent and can be characterised by a non-homogeneous Poisson process. In this paper we develop and analyze a two-node tandem queueing model with the assumption that the arrivals follows a non-homogeneous compound Poisson process, and the service rates of each node depends on the number of customers in the queue connected it. Using the difference - differential equations and a probability generating function of the number of customers in the queue and in the system is analysed. Explicit expression are derived for performance measures including average number of customers in each queue, the probability of emptiness of the system, the average waiting time in the queue and in the system, the throughput of the nodes, and the variance of the number of customers in the queue. Sensitivity analysis of the model with respect to the uniformly distributed batch size arrivals is also studied. It is observed that the system performance measures are highly influenced by non-homogeneous (time-dependent) arrivals rate and batch size distribution parameters. This model also includes several earlier queuing models as particular cases for specific values of the parameters. These models are useful in scheduling tele and wireless communication networks.

Keywords: Non- Homogeneous Poisson process, Bulk arrivals, State-dependent service rates, Tandem Queueing models.

## 1. INTRODUCTION :
Queueing models are essential for designing and monitoring of several communication systems. The queueing models provide the optimum operating policies of the several systems. They are essential for performance evaluation of a wide variety of systems in communication networks, production manufacturing process, data/voice transmission, nero physiological systems, cargo handling, and ATM scheduling. [1]. Starting from the first queueing model by A.K. Erlang [37] much work has been reported in literature regarding queueing models and their applications. These models are formed with suitable assumptions on arrival, service processes. In many of the practical situations the output from one queueing process serves as input to other i.e., the queues are connected in series. These types of queueing systems are called tandem

queueing systems. O' Brien [2], Jackson [3,5], Paul [4] have pioneered the tandem queueing models. Recently Yukuo Hayashida [6], Srinivasa Rao et al.[7], Yongjian et al.[8], Lieshout and Mandjes [9], Wai Kin Victor Chan [10], Che soong Kin et al. [11,12] and others have studies various tandem queueing models. In all these papers they considered that the arrival and service processes are independent. In many practical situations like communication networks the service rate is to be may dependent on the number of customers in the queue. This type of queueing models are called as queueing models with state-dependent service rates or queueing models with load dependent. To have accurate prediction of the performance measures of the system and to obtain optimal policies the load dependent queueing models are developed and analysed. [13-18]. In all these papers the authors assumed that the arrivals to the system come singly and can be characterised by a Poisson process.

In some practical situations like stored and forward communication networks the arrivals cannot be characterized by a Poisson process the arriving messages are converted into a random number of packets depending upon the size of the message. Similarly, in railway yards and ports the cargo handling is done in batches of random size. For analyzing these sort of situations the queueing models with bulk arrivals are developed. The bulk arrivals can be well characterised by a compound Poisson process. The compound Poisson process also includes Poisson process as a particular case. The bulk arrival queueing models were initiated by Erlang Solution of M/Ek/1 model Brockmeyer et al [19]. Later, several authors developed various queueing models for bulk arrivals. Recently, Madan et al [20] studied a single-server queue with batch arrivals having two types of heterogeneous service with a different service rates having general service time distribution. Chaudhry and Chang [21] developed a discrete time bulk–service queueing model known as the Geo/$G^Y$/1/N+B model. Juan [22] developed M/$G^Y$/1 a queueing model, with a discretised service time distribution. Schleyer and Furmans [23] presented an analytical method to calculate the waiting time distribution for the G/G/1 queueing system with batch arrivals. Ahmed [24] considered a multi-channel bi-level heterogeneous-server bulk arrival queueing system with an Erlangian service time. Chen A, Pollett P., LiJ., and Zhang H [25] studied a modified Markovian bulk arrival and bulk service queue incorporating state-dependent control. Charan Jeet Singh et al. (2011) studied a single-server bulk queueing system with state dependent rates and a second optional service. They used the supplementary variable technique to obtain the probability generating function. Dieter Claeys et al. [26] studied a threshold-based service system with batch arrivals and general service times. Arumuganathan et al. [27] studied two-node tandem communication network models with bulk arrivals using queueing theory. In all these models, the authors considered the arrivals to be homogeneous and follow compound Poisson process.

However, in queueing models connected with communication networks like Ethernet LAN, metropolitan area network (MAN) traffic, wider area network (WAN), and variable bit rate (VBR) traffic exhibit time-dependent arrival rates and cannot be modelled with homogeneous Poisson process or compound Poisson process [28-31]. The studies made by Crovella et al. [32], Murali Krishna et al. [33], Feldmann A. [34], Fischer et al. [35] also revealed that in a TCP connect communication network, the time between packet arrivals cannot be characterized by an exponential distribution. The observations made by Dinda et al. [36] have revealed that the traffic generated by many real world applications exhibits a high degree of burstiness (time varying arrival rate). Very little work has been reported regarding tandem queueing models with non homogeneous bulk arrivals. For this purpose, in this paper, we develop and analyse a two-node tandem queueing model with the assumption that the arrival process follows a non-homogeneous compound Poisson process having state-dependent service rates. The non-homogeneous compound Poisson process is capable of including a wide spectrum of Poisson process and can accommodate heterogeneous or brustiness and non smooth traffic arrivals.

Using difference - differential equations the joint probability generating function of the number of customers in each queue is derived. The system performance measures like the average number of customers in the queue and system, the average waiting time of the customers in the system and queue, the throughput of the node, etc., are derived. The sensitivity of the model with respect to the parameters is also studies by assuming that the batch size of the arrivals follows a uniform distribution. A comparative study of the developed model with that of homogeneous and non homogenous (time dependent) arrivals is also discussed.

## 2. QUEUING MODEL:

In this section, we consider two queues $Q_1$, $Q_2$ and service stations $S_1$, $S_2$ which are connected as series in order. It is assumed that the customers after getting service through first service station may join the second queue which is in series connected to $S_1$. It is further assumed that the customers arrive to the first queue in batches of random size but dependent on time i.e., the actual number of customers in any arriving module is a random variable X with probability $C_x$. In other words, the arrival of customers follows non-homogeneous compound Poisson processes with a mean composite arrival rate $\lambda(t)$ having bulk size distributions $C_x$. The service completion in both the service stations follows Poisson processes with the parameters $\mu_1$ and $\mu_2$ for the first and second service stations. It is further assumed that the mean service rate in the service station is linearly dependent on the content of the queue connected to it. The queue discipline is first in, first out. A schematic diagram representing the queueing model is shown in figure 2.1

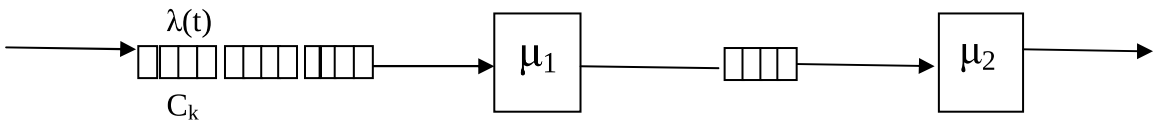



**Figure 2.1** Schematic diagram of the queueing model

Let $n_1$ and $n_2$ denote the number of customers in first and second queues and let $P_{n_1,n_2}(t)$ be the probability that there are $n_1$ customers in the first queue and $n_2$ customers in the second queue at time t. The difference - differential equations governing the model are as follows:

$$\frac{\partial P_{n_1,n_2}(t)}{\partial t} = -(\lambda(t) + n_1\mu_1 + n_2\mu_2)P_{n_1,n_2}(t) + \lambda(t)\sum_{k=1}^{n_1} P_{n_1-k,n_2}(t)C_k + (n_1+1)\mu_1 P_{n_1+1,n_2-1}(t) + (n_2+1)\mu_2 P_{n_1,n_2+1}(t), \quad for\ n_1, n_2 > 0$$

$$\frac{\partial P_{n_1,0}(t)}{\partial t} = -(\lambda(t) + n_1\mu_1)P_{n_1,0}(t) + \lambda(t)\left[\sum_{k=1}^{n_1} P_{n_1-k,0}(t)C_k\right] + \mu_2 P_{n_1,1}(t), for\ n_1 > 0, n_2 = 0$$

$$\frac{\partial P_{0,n_2}(t)}{\partial t} = -(\lambda(t) + n_2\mu_2)P_{0,n_2}(t) + \mu_1 P_{1,n_2-1}(t) + (n_2+1)\mu_2 P_{0,n_2+1}(t), for\ n_2 > 0, n_1 = 0$$

$$\frac{\partial P_{0,0}(t)}{\partial t} = -\lambda(t)P_{0,0}(t) + \mu_2 P_{0,1}(t), \qquad for\ n_1, n_2 = 0$$

$$\frac{\partial P_{1,0}(t)}{\partial t} = -(\lambda(t) + \mu_1)P_{1,0}(t) + \mu_2 P_{1,1}(t) + \lambda(t)P_{0,0}(t)C_1, \quad for\ n_1 = 1, n_2 = 0$$

$$\frac{\partial P_{0,1}(t)}{\partial t} = -(\lambda(t) + \mu_2)P_{0,1}(t) + \mu_1 P_{1,0}(t) + 2\mu_2 P_{0,2}(t), \quad for\ n_1 = 0, n_2 = 1 \tag{1}$$

Let the joint probability generating function of $P_{n_1,n_2}(t)$ is be

$$P(Z_1,Z_2,t) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Z_1^{n_1} Z_2^{n_2} P_{n_1,n_2}(t)$$

(2)

Multiplying the equation (2.1) by $Z_1$, $Z_2$ and summing over all $n_1$and $n_2$, we obtain

$$\frac{\partial P(Z_1,Z_2,t)}{\partial t} = [-\mu_1(Z_1 - Z_2)]\frac{\partial P(Z_1,Z_2,t)}{\partial Z_1} + [\mu_2(Z_2-1)]\frac{\partial P(Z_1,Z_2,t)}{\partial Z_2} + P(Z_1,Z_2,t)[\lambda(t)(C(Z)-1)]$$

where $C(Z) = \sum_{k=1}^{\infty} Z^k C_k$ is the probability generating function for the arriving batch size distribution of $Q_1$.

(3)

Solving the equation (3) by the Lagrangian method, the auxiliary equations are

$$\frac{\partial t}{1} = \frac{\partial Z_1}{\mu_1(Z_1 - Z_2)} = \frac{\partial Z_2}{\mu_2(Z_2-1)} = \frac{\partial P(Z_1,Z_2,t)}{P(Z_1,Z_2,t)[\lambda(t)(C(Z)-1)]}$$

(4)

Solving the first and third terms in equation (4), we obtain

$$a = (Z_2 - 1)e^{-\mu_2 t}$$

Solving the first and second terms in equation (4), we obtain

$$b = (Z_1 - 1)e^{-\mu_1 t} + a\mu_1 \frac{e^{(\mu_2-\mu_1)t}}{(\mu_2-\mu_1)}$$

Solving the first and fifth terms in equation (4), we obtain

$$c = P(Z_1,Z_2,t)\exp\left\{\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^i C_k\binom{k}{r}\binom{r}{i} b^{r-i}\left[\frac{a\mu_1}{\mu_2-\mu_1}\right]^i\left[\frac{-\lambda}{i\mu_2+(r-i)\mu_1} + \left[\frac{a}{[i\mu_2+(r-i)\mu_1]^2}\right]\right]\right\}$$

where a, b and c are arbitrary constants derived using the initial conditions $P_{00}(0) = 1, P_{00}(t) = 0\ \forall, t > 0$

(5)

The general solution of (2.4) gives the probability generating function of the number of customers in the first queue and the number of customers in the second queue at time ‘t’ as

$$\begin{aligned}P(Z_1,Z_2,t) = \exp\Bigg\{&\lambda\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^i C_k\binom{k}{r}\binom{r}{i}\left[\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^i\Bigg[(Z_1-1)\\ &+\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\Bigg]^{r-i}\left[\frac{1-e^{-[i\mu_2+(r-i)\mu_1]t}}{i\mu_2+(r-i)\mu_1}\right]\\ &+\alpha\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^i C_k\binom{k}{r}\binom{r}{i}\left[\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^i\Bigg[(Z_1-1)\\ &+\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\Bigg]^{r-i}\left[\frac{t[i\mu_2+(r-i)\mu_1]-1+e^{-[i\mu_2+(r-i)\mu_1]t}}{[i\mu_2+(r-i)\mu_1]^2}\right]\Bigg\}, |Z_1| < 1, |Z_2| < 1\end{aligned}$$

(6)

## 3. CHARACTERISTICS OF THE QUEUING MODEL:

Expanding $P(Z_1, Z_2, t)$ given in equation (6) and collecting the constant terms, we obtain the probability that the system is empty to be

$$P_{00}(t)$$
$$= \exp\left\{\lambda \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^{i+r} C_k \binom{k}{r}\binom{r}{i}\left[\frac{\mu_2^{\,r-i}\mu_1^{\,i}}{(\mu_2-\mu_1)^r}\right]\left[\frac{1-e^{-[i\mu_2+(r-i)\mu_1]t}}{i\mu_2+(r-i)\mu_1}\right]\right.$$
$$\left.+\,\alpha \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^{i+r} C_k \binom{k}{r}\binom{r}{i}\left[\frac{\mu_2^{\,r-i}\mu_1^{\,i}}{(\mu_2-\mu_1)^r}\right]\left[\frac{t[i\mu_2+(r-i)\mu_1]-1+e^{-[i\mu_2+(r-i)\mu_1]t}}{[i\mu_2+(r-i)\mu_1]^2}\right]\right\}$$
(7)

Taking $Z_2=1$ in $P(Z_1, Z_2, t)$, we determine probability generating function for the first queue size to be

$$P(Z_1,t) = exp\left\{\lambda \sum_{k=1}^{\infty}\sum_{r=1}^{k} C_k \binom{k}{r}(Z_1-1)^r\left[\frac{1-e^{-r\mu_1 t}}{r\mu_1}\right]\right.$$
$$\left.+\,\alpha \sum_{k=1}^{\infty}\sum_{r=1}^{k} C_k \binom{k}{r}(Z_1-1)^r\left[\frac{tr\mu_1-1+e^{-r\mu_1 t}}{[r\mu_1]^2}\right]\right\}$$
(8)

By expanding $P(Z_1,t)$ and collecting the constant terms, we determine the probability that the first queue is empty to be

$P_{0.}(t) = exp\{B_1(t)\}$ say

$$\text{where } B_1(t) = \lambda \sum_{k=1}^{\infty}\sum_{r=1}^{k} C_k \binom{k}{r}(-1)^r\left[\frac{1-e^{-r\mu_1 t}}{r\mu_1}\right] + \alpha \sum_{k=1}^{\infty}\sum_{r=1}^{k} C_k \binom{k}{r}(-1)^r\left[\frac{tr\mu_1-1+e^{-r\mu_1 t}}{[r\mu_1]^2}\right]$$
(9)

The mean number of customers in the first queue is

$$L_1(t) = E(X)\left[\frac{\lambda}{\mu_1}[1-e^{-\mu_1 t}] + \frac{\alpha}{\mu_1^{\,2}}[t\mu_1-1+e^{-\mu_1 t}]\right]$$

where E(X) is the mean of the batch size arrivals. (10)

The utilisation of the first service station is

$$U_1(t) = 1 - exp\{B_1(t)\} \tag{11}$$

The throughput of the service station is

$$Th_1(t) = \mu_1[1 - exp\{B_1(t)\}] \tag{12}$$

The average waiting time of a customer in the first queue is

$$W_1(t) = \frac{E(X)\left[\frac{\lambda}{\mu_1}[1-e^{-\mu_1 t}] + \frac{\alpha}{\mu_1^{\,2}}[t\mu_1-1+e^{-\mu_1 t}]\right]}{\mu_1[1-exp\{B_1(t)\}]}$$
(13)

The variance of the number of customers in the first queue is

$$V_1(t) = E(X)\left[\frac{\lambda}{\mu_1}\left[(k-1)\left(\frac{1-e^{-2\mu_1 t}}{2}\right)+(1-e^{-\mu_1 t})\right] + \frac{\alpha}{{\mu_1}^2}\left[(k-1)\left(\frac{2\mu_1 t-1+e^{-2\mu_1 t}}{4}\right)+(t\mu_1-1+e^{-\mu_1 t})\right]\right] \tag{14}$$

The coefficient of variation of the number of customers in the first queue is

$$CV_1(t) = \frac{\sqrt{V_1(t)}}{L_1(t)} X100 \tag{15}$$

Taking $Z_1 = 1$ in $P(Z_1, Z_2, t)$, we determine probability generating function for the second queue size to be

$$P(Z_2,t) = exp\left\{\lambda\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^i C_k\binom{k}{r}\binom{r}{i}\left[\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^r\left[\frac{1-e^{-[i\mu_2+(r-i)\mu_1]t}}{i\mu_2+(r-i)\mu_1}\right] + \alpha\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^i C_k\binom{k}{r}\binom{r}{i}\left[\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^r\left[\frac{t[i\mu_2+(r-i)\mu_1]-1+e^{-[i\mu_2+(r-i)\mu_1]t}}{[i\mu_2+(r-i)\mu_1]^2}\right]\right\} \tag{16}$$

By expanding $P(Z_2,t)$ and collecting the constant terms, we obtain the probability that the second queue is empty to be

$P_{.0}(t) = exp\{B_2(t)\}$ say

$$B_2(t) = \lambda\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^{i+r} C_k\binom{k}{r}\binom{r}{i}\left[\frac{\mu_1}{\mu_2-\mu_1}\right]^r\left[\frac{1-e^{-[i\mu_2+(r-i)\mu_1]t}}{i\mu_2+(r-i)\mu_1}\right] + \alpha\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^{i+r} C_k\binom{k}{r}\binom{r}{i}\left[\frac{\mu_1}{\mu_2-\mu_1}\right]^r\left[\frac{t[i\mu_2+(r-i)\mu_1]-1+e^{-[i\mu_2+(r-i)\mu_1]t}}{[i\mu_2+(r-i)\mu_1]^2}\right] \tag{17}$$

The mean number of customers in the second queue is

$$L_2(t) = E(X)\left[\frac{\mu_1}{\mu_2-\mu_1}\right]\left[\frac{\lambda}{\mu_1}[1-e^{-\mu_1 t}] + \frac{\alpha}{{\mu_1}^2}[t\mu_1-1+e^{-\mu_1 t}]\right]$$

where E(X) is the mean of the batch size arrivals. (18)

The utilisation of the second service station is

$$U_2(t) = 1 - exp\{B_2(t)\} \tag{19}$$

The throughput of the service station is

$$Th_2(t) = \mu_2[1 - exp\{B_2(t)\}] \tag{20}$$

The average waiting time of a customer in the second queue is

$$W_2(t) = \frac{E(X)\left[\frac{\mu_1}{\mu_2 - \mu_1}\right]\left[\frac{\lambda}{\mu_1}[1 - e^{-\mu_1 t}] + \frac{\alpha}{\mu_1{}^2}[t\mu_1 - 1 + e^{-\mu_1 t}]\right]}{\mu_2[1 - \exp\{B_2(t)\}]} \tag{21}$$

The variance of the number of customers in the second queue is

$$\begin{aligned} V_2(t) = E(X)\frac{(k-1)}{2}\left[\frac{\mu_1}{\mu_2 - \mu_1}\right]^2 \Bigg(\lambda\left[\left(\frac{1 - e^{-2\mu_1 t}}{\mu_1}\right) - 4\left(\frac{1 - e^{-(\mu_2+\mu_1)t}}{\mu_2 + \mu_1}\right) + \left(\frac{1 - e^{-2\mu_2 t}}{\mu_2}\right)\right] \\ + \alpha\left[\left(\frac{2\mu_1 t - 1 + e^{-2\mu_1 t}}{2\mu_1{}^2}\right) - 4\left(\frac{t[\mu_2 + \mu_1] - 1 + e^{-[\mu_2+\mu_1]t}}{[\mu_2 + \mu_1]^2}\right) \right. \\ \left. + \left(\frac{2\mu_2 t - 1 + e^{-2\mu_2 t}}{2\mu_2{}^2}\right)\right]\Bigg) \\ + E(X)\left[\frac{\mu_1}{\mu_2 - \mu_1}\right]\left[\frac{\lambda}{\mu_1}[1 - e^{-\mu_1 t}] + \frac{\alpha}{\mu_1{}^2}[t\mu_1 - 1 + e^{-\mu_1 t}]\right] \end{aligned} \tag{22}$$

The coefficient of variation of the number of customers in the second queue is

$$CV_2(t) = \frac{\sqrt{V_2(t)}}{L_2(t)} X100 \tag{23}$$

The mean number of customers in the queueing system at time t is

$$L(t) = L_1(t) + L_2(t) \tag{24}$$

where, $L_1(t)$, $L_2(t)$ are as given equations (10) and (18).

## 4. CHARACTERISTICS OF THE MODEL UNDER UNIFORM BATCH SIZE DISTRIBUTION

The performance measures of the queueing model are highly influenced by the form of the batch size distribution. It is assumed that the number of customers in any arriving module is random and follows a uniform distribution with parameters a and b. Then, the probability mass function of the batch size distributions is $C_k = \frac{1}{b-a+1}$ $for\ k = a, a+1, \dots b$ $Q_1$ and $Q_2$ respectively.

The mean number of customers in the batch is

$$E(X) = \frac{a+b}{2}$$

The variance of the batch size is

$$V(X) = \frac{1}{12}[(b - a + 1)^2 - 1]$$

The joint probability generating function for the number of the customers in each queue is

$$P(Z_1,Z_2,t) = \exp\left\{\lambda \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^i \frac{1}{b-a+1}\binom{k}{r}\binom{r}{i}\left[\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^i\left[(Z_1-1) + \frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^{r-i}\left[\frac{1-e^{-[i\mu_2+(r-i)\mu_1]t}}{i\mu_2+(r-i)\mu_1}\right] + \alpha \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^i \frac{1}{b-a+1}\binom{k}{r}\binom{r}{i}\left[\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^i\left[(Z_1-1) + \frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^{r-i}\left[\frac{t[i\mu_2+(r-i)\mu_1]-1+e^{-[i\mu_2+(r-i)\mu_1]t}}{[i\mu_2+(r-i)\mu_1]^2}\right]\right\}, |Z_1|<1, |Z_2|<1 \tag{25}$$

Taking $Z_1 = Z_2 = 0$ in $P(Z_1, Z_2, t)$, we determine the probability generating function for the queueing system is empty to be

$$P_{00}(t) = \exp\left\{\lambda \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^{i+r}\frac{1}{b-a+1}\binom{k}{r}\binom{r}{i}\left[\frac{\mu_2{}^{r-i}\mu_1{}^i}{(\mu_2-\mu_1)^r}\right]\left[\frac{1-e^{-[i\mu_2+(r-i)\mu_1]t}}{i\mu_2+(r-i)\mu_1}\right] + \alpha \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^{i+r}\frac{1}{b-a+1}\binom{k}{r}\binom{r}{i}\left[\frac{\mu_2{}^{r-i}\mu_1{}^i}{(\mu_2-\mu_1)^r}\right]\left[\frac{t[i\mu_2+(r-i)\mu_1]-1+e^{-[i\mu_2+(r-i)\mu_1]t}}{[i\mu_2+(r-i)\mu_1]^2}\right]\right\} \tag{26}$$

Taking $Z_2 = 1$ in $P(Z_1, Z_2, t)$, we determine probability generating function for the first queue size to be

$$P(Z_1,t) = exp\left\{\lambda \sum_{k=1}^{\infty}\sum_{r=1}^{k}\frac{1}{b-a+1}\binom{k}{r}(Z_1-1)^r\left[\frac{1-e^{-r\mu_1 t}}{r\mu_1}\right] + \alpha \sum_{k=1}^{\infty}\sum_{r=1}^{k}\frac{1}{b-a+1}\binom{k}{r}(Z_1-1)^r\left[\frac{tr\mu_1-1+e^{-r\mu_1 t}}{[r\mu_1]^2}\right]\right\} \tag{27}$$

By expanding $P(Z_1,t)$ and collecting the constant terms, we determine the probability that the first queue is empty to be

$P_{0.}(t) = exp\{B_3(t)\}$ say

$$\text{where } B_3(t) = \lambda \sum_{k=1}^{\infty}\sum_{r=1}^{k}\frac{1}{b-a+1}\binom{k}{r}(-1)^r\left[\frac{1-e^{-r\mu_1 t}}{r\mu_1}\right] + \alpha \sum_{k=1}^{\infty}\sum_{r=1}^{k}\frac{1}{b-a+1}\binom{k}{r}(-1)^r\left[\frac{tr\mu_1-1+e^{-r\mu_1 t}}{[r\mu_1]^2}\right] \tag{28}$$

The mean number of customers in the first queue is

$$L_1(t) = \frac{a+b}{2}\left[\frac{\lambda}{\mu_1}[1-e^{-\mu_1 t}] + \frac{\alpha}{{\mu_1}^2}[t\mu_1 - 1 + e^{-\mu_1 t}]\right] \tag{29}$$

The utilisation of the first service station is

$$U_1(t) \quad = 1 - exp\{B_3(t)\} \tag{30}$$

The throughput of the service station is

$$Th_1(t) = \mu_1\,[1 - exp\{B_3(t)\}] \tag{31}$$

The average waiting time of a customer in the first queue is

$$W_1(t) = \frac{\frac{a+b}{2}\left[\frac{\lambda}{\mu_1}[1-e^{-\mu_1 t}] + \frac{\alpha}{{\mu_1}^2}[t\mu_1 - 1 + e^{-\mu_1 t}]\right]}{\mu_1\,[1 - exp\{B_3(t)\}]} \tag{32}$$

The variance of the number of customers in the first queue is

$$V_1(t) = \frac{a+b}{2}\left[\frac{\lambda}{\mu_1}\left[(k-1)\left(\frac{1-e^{-2\mu_1 t}}{2}\right) + (1-e^{-\mu_1 t})\right]\right.$$
$$\left. + \frac{\alpha}{{\mu_1}^2}\left[(k-1)\left(\frac{2\mu_1 t - 1 + e^{-2\mu_1 t}}{4}\right) + (t\mu_1 - 1 + e^{-\mu_1 t})\right]\right] \tag{33}$$

The coefficient of variation of the number of customers in the first queue is

$$CV_1(t) = \frac{\sqrt{V_1(t)}}{L_1(t)} X100 \tag{34}$$

Taking $Z_1 = 1$ in $P(Z_1, Z_2, t)$, we determine probability generating function for the second queue size to be

$$P(Z_2,t)$$
$$= exp\left\{\lambda \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^i \frac{1}{b-a+1}\binom{k}{r}\binom{r}{i}\left[\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^r\left[\frac{1-e^{-[i\mu_2+(r-i)\mu_1]t}}{i\mu_2+(r-i)\mu_1}\right]\right.$$
$$\left. + \alpha \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^i \frac{1}{b-a+1}\binom{k}{r}\binom{r}{i}\left[\frac{(Z_2-1)\mu_1}{\mu_2-\mu_1}\right]^r\left[\frac{t[i\mu_2+(r-i)\mu_1] - 1 + e^{-[i\mu_2+(r-i)\mu_1]t}}{[i\mu_2+(r-i)\mu_1]^2}\right]\right\} \tag{35}$$

By expanding $P(Z_2,t)$ and collecting the constant terms, we determine the probability that the second queue is empty to be

$P_{.0}(t) = exp\{B_4(t)\}$ say

$$\begin{aligned} where\ B_4(t) &= \lambda \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^{i+r}\frac{1}{b-a+1}\binom{k}{r}\binom{r}{i}\left[\frac{\mu_1}{\mu_2-\mu_1}\right]^r\left[\frac{1-e^{-[i\mu_2+(r-i)\mu_1]t}}{i\mu_2+(r-i)\mu_1}\right] \\ &+ \alpha \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{i=0}^{r}(-1)^{i+r}\frac{1}{b-a+1}\binom{k}{r}\binom{r}{i}\left[\frac{\mu_1}{\mu_2-\mu_1}\right]^r\left[\frac{t[i\mu_2+(r-i)\mu_1]-1+e^{-[i\mu_2+(r-i)\mu_1]t}}{[i\mu_2+(r-i)\mu_1]^2}\right] \end{aligned} \tag{36}$$

The mean number of customers in the second queue is

$$L_2(t) = \frac{a+b}{2}\left[\frac{\mu_1}{\mu_2-\mu_1}\right]\left[\frac{\lambda}{\mu_1}[1-e^{-\mu_1 t}] + \frac{\alpha}{\mu_1^2}[t\mu_1 - 1 + e^{-\mu_1 t}]\right] \tag{37}$$

The utilisation of the second service station is

$$U_2(t) = 1 - exp\{B_4(t)\} \tag{38}$$

The throughput of the service station is

$$Th_2(t) = \mu_2[1 - exp\{B_4(t)\}] \tag{39}$$

The average waiting time of a customer in the second queue is

$$W_2(t) = \frac{\frac{a+b}{2}\left[\frac{\mu_1}{\mu_2-\mu_1}\right]\left[\frac{\lambda}{\mu_1}[1-e^{-\mu_1 t}] + \frac{\alpha}{\mu_1^2}[t\mu_1 - 1 + e^{-\mu_1 t}]\right]}{\mu_2[1 - exp\{B_4(t)\}]} \tag{40}$$

The variance of the number of customers in the second queue is

$$\begin{aligned} V_2(t) = \frac{a+b}{2}\frac{(k-1)}{2}\left[\frac{\mu_1}{\mu_2-\mu_1}\right]^2 &\left(\lambda\left[\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) - 4\left(\frac{1-e^{-(\mu_2+\mu_1)t}}{\mu_2+\mu_1}\right) + \left(\frac{1-e^{-2\mu_2 t}}{\mu_2}\right)\right]\right. \\ &+ \alpha\left[\left(\frac{2\mu_1 t - 1 + e^{-2\mu_1 t}}{2\mu_1^2}\right) - 4\left(\frac{t[\mu_2+\mu_1] - 1 + e^{-[\mu_2+\mu_1]t}}{[\mu_2+\mu_1]^2}\right)\right. \\ &\left.\left.+ \left(\frac{2\mu_2 t - 1 + e^{-2\mu_2 t}}{2\mu_2^2}\right)\right]\right) \\ &+ \frac{a+b}{2}\left[\frac{\mu_1}{\mu_2-\mu_1}\right]\left[\frac{\lambda}{\mu_1}[1-e^{-\mu_1 t}] + \frac{\alpha}{\mu_1^2}[t\mu_1 - 1 + e^{-\mu_1 t}]\right] \end{aligned} \tag{41}$$

The coefficient of variation of the number of customers in the second queue is

$$CV_2(t) = \frac{\sqrt{V_2(t)}}{L_2(t)} X100 \tag{42}$$

The mean number of customers in the queueing system at time t is

$$L(t) = L_1(t) + L_2(t)$$

where $L_1(t)$, $L_2(t)$ are as given equations (29),(37). (43)

## 5. NUMERICAL DEMONSTRATION AND SENSITIVITY ANALYSIS:

In this section, the performance of the proposed queueing model is discussed through a numerical illustration. The customers arrive in batches to the first queue and after getting service though the first server join the second queue, which is serially connected to the first service station. The arrival of the customers follows a compound Poisson process the composite mean arrival rate is $\lambda(t) = \lambda + \alpha t$. Each arriving module represents a batch of customers. The number of customers in each arriving module follows a uniform distribution with parameters (a,b). Because the characteristics of the queueing model are highly sensitive with respect to time, the transient behaviour of the model is studied by computing the performance measures with the following set of values for the model parameters:

$t$ = 0.05, 0.06, 0.07, 0.08, 0.09; $a$ = 2, 3, 4, 5, 6; $b$ = 12, 14, 16, 18, 20; $\lambda$ = 1, 1.5, 2, 2.5, 3;
$\alpha$ = 0, 1, 1.5, 2, 2.5; $\mu_1$ = 16, 17, 18, 19, 20; $\mu_2$ = 22, 23, 24, 25, 26;

From the equations of the probability that the system is empty and the emptiness of the marginal queues, the expected number of customers, and the utilisation of servers are computed for different values of the parameters t, a, b, $\lambda$, $\alpha$, $\mu_1$, $\mu_2$, and presented in Table 1. The relationships between the parameters and performance measures are shown in figure 2.

From Table 1, it can be observed that the probability of emptiness of the queueing system and two marginal queues are highly sensitive with respect to time. As time (t) increases from 0.05 to 0.09 the probability that the queue is empty decreases; the probability that the marginal queues is empty decreases; the expected number of customers in each queue and in the system increases; and the utilisation of the service station increases, when all other parameters are fixed.

As the parameter a increases from 2 to 6, the probability that the queue is empty unchanged; the probability that the marginal queues is empty decreases; the expected number of customers in each queue and in the system increases; and the utilisation of the service station increases, when all other parameters are fixed. As the parameter b increases from 12 to 20, the probability that the queue is empty decreases; the probability that the marginal queues is empty decreases; the expected number of customers in each queue and in the system increases; and the utilisation of the service station increases, when all other parameters are fixed.

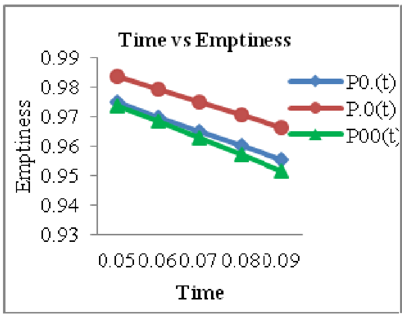
It can be further observed that as the parameter ($\lambda$) increases from 1 to 3, the probability that the queue is empty decreases; the probability that the marginal queue is empty decreases; the expected number of customers in each queue and in the system increases; and the utilisation of the service station increases, when all other parameters are fixed. The same patterns hold with respect to the parameter ($\alpha$). It can also be observed that as the service rate ($\mu_1$) increases from 16 to 20, the probability that the queue is empty remains constant; the probability that the first queue increases, but the second queue decreases; the expected number of customers in the first queue decreases, but the second queue and the system increases; and the utilisation of the service station in the first queue decreases, but the second queue increases, when all other parameters are fixed. It can also be observed that as the service rate ($\mu_2$) increases from 22 to 26, the probability that the queue is empty increases and remains constant; the probability that the second queue increases, but the first queue it remains constant; the expected number of customers in the queue, and the utilisation of the service station in the second queue decreases, but the first queue it remains constant.

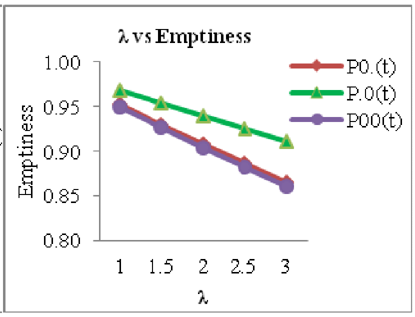
**Table – 1**

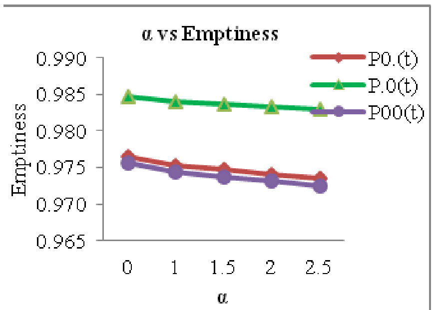
Values of $P_{0.}(t)$, $P_{.0}(t)$, $P_{00}(t)$, $L_1(t)$, $L_2(t)$, $L(t)$, $U_1(t)$, $U_2(t)$ for different values of parameters

| t | a | b | λ | α | $\mu_1$ | $\mu_2$ | $P_{0.}(t)$ | $P_{.0}(t)$ | $P_{00}(t)$ | $L_1(t)$ | $L_2(t)$ | $L(t)$ | $U_1(t)$ | $U_2(t)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 1 | 10 | 0.5 | 1.5 | 15 | 21 | 0.9747 | 0.9837 | 0.9737 | 0.1049 | 0.2622 | 0.3671 | 0.0253 | 0.0163 |
| 0.06 | | | | | | | 0.9697 | 0.9793 | 0.9683 | 0.1200 | 0.3001 | 0.4201 | 0.0303 | 0.0207 |
| 0.07 | | | | | | | 0.9648 | 0.9749 | 0.9627 | 0.1338 | 0.3346 | 0.4684 | 0.0352 | 0.0251 |
| 0.08 | | | | | | | 0.9600 | 0.9704 | 0.9572 | 0.1465 | 0.3662 | 0.5127 | 0.0400 | 0.0296 |
| 0.09 | | | | | | | 0.9553 | 0.9660 | 0.9516 | 0.1581 | 0.3954 | 0.5535 | 0.0447 | 0.0340 |
| | 2 | | | | | | 0.9740 | 0.9825 | 0.9735 | 0.1144 | 0.2861 | 0.4005 | 0.0260 | 0.0175 |
| | 3 | | | | | | 0.9737 | 0.9815 | 0.9735 | 0.1240 | 0.3099 | 0.4339 | 0.0263 | 0.0185 |
| | 4 | | | | | | 0.9736 | 0.9807 | 0.9735 | 0.1335 | 0.3337 | 0.4672 | 0.0264 | 0.0193 |
| | 5 | | | | | | 0.9736 | 0.9801 | 0.9735 | 0.1430 | 0.3576 | 0.5006 | 0.0264 | 0.0199 |
| | 6 | | | | | | 0.9735 | 0.9795 | 0.9735 | 0.1526 | 0.3814 | 0.5340 | 0.0265 | 0.0205 |
| | | 12 | | | | | 0.9745 | 0.9826 | 0.9737 | 0.1240 | 0.3099 | 0.4339 | 0.0255 | 0.0174 |
| | | 14 | | | | | 0.9744 | 0.9818 | 0.9737 | 0.1430 | 0.3576 | 0.5006 | 0.0256 | 0.0182 |
| | | 16 | | | | | 0.9743 | 0.9811 | 0.9736 | 0.1621 | 0.4052 | 0.5673 | 0.0257 | 0.0189 |
| | | 18 | | | | | 0.9742 | 0.9805 | 0.9736 | 0.1812 | 0.4529 | 0.6341 | 0.0258 | 0.0195 |
| | | 20 | | | | | 0.9741 | 0.9800 | 0.9736 | 0.2002 | 0.5006 | 0.7008 | 0.0259 | 0.0200 |
| | | | 1 | | | | 0.9518 | 0.9685 | 0.9499 | 0.2016 | 0.5040 | 0.7056 | 0.0482 | 0.0315 |
| | | | 1.5 | | | | 0.9295 | 0.9537 | 0.9267 | 0.2984 | 0.7459 | 1.0443 | 0.0705 | 0.0463 |
| | | | 2 | | | | 0.9076 | 0.9390 | 0.9041 | 0.3951 | 0.9877 | 1.3828 | 0.0924 | 0.0610 |
| | | | 2.5 | | | | 0.8863 | 0.9246 | 0.8820 | 0.4918 | 1.2295 | 1.7213 | 0.1137 | 0.0754 |
| | | | 3 | | | | 0.8655 | 0.9104 | 0.8604 | 0.5886 | 1.4714 | 2.0600 | 0.1345 | 0.0896 |
| | | | | 0 | | | 0.9765 | 0.9846 | 0.9756 | 0.0967 | 0.2418 | 0.3385 | 0.0235 | 0.0154 |
| | | | | 1 | | | 0.9753 | 0.9840 | 0.9743 | 0.1022 | 0.2554 | 0.3576 | 0.0247 | 0.0160 |
| | | | | 1.5 | | | 0.9747 | 0.9837 | 0.9737 | 0.1049 | 0.2622 | 0.3671 | 0.0253 | 0.0163 |
| | | | | 2 | | | 0.9741 | 0.9833 | 0.9731 | 0.1076 | 0.2690 | 0.3766 | 0.0259 | 0.0167 |
| | | | | 2.5 | | | 0.9735 | 0.9830 | 0.9725 | 0.1103 | 0.2758 | 0.3861 | 0.0265 | 0.0170 |
| | | | | | 16 | | 0.9748 | 0.9833 | 0.9738 | 0.1027 | 0.3286 | 0.4313 | 0.0252 | 0.0167 |
| | | | | | 17 | | 0.9749 | 0.9829 | 0.9738 | 0.1005 | 0.4273 | 0.5278 | 0.0251 | 0.0171 |
| | | | | | 18 | | 0.9751 | 0.9825 | 0.9738 | 0.0985 | 0.5908 | 0.6893 | 0.0249 | 0.0175 |
| | | | | | 19 | | 0.9752 | 0.9822 | 0.9738 | 0.0965 | 0.9163 | 1.0128 | 0.0248 | 0.0178 |
| | | | | | 20 | | 0.9753 | 0.9818 | 0.9738 | 0.0945 | 1.8901 | 1.9846 | 0.0247 | 0.0182 |
| | | | | | | 22 | 0.9747 | 0.9838 | 0.9737 | 0.1049 | 0.2248 | 0.3297 | 0.0253 | 0.0162 |
| | | | | | | 23 | 0.9747 | 0.9839 | 0.9738 | 0.1049 | 0.1967 | 0.3016 | 0.0253 | 0.0161 |
| | | | | | | 24 | 0.9747 | 0.9840 | 0.9738 | 0.1049 | 0.1748 | 0.2797 | 0.0253 | 0.0160 |
| | | | | | | 25 | 0.9747 | 0.9841 | 0.9738 | 0.1049 | 0.1573 | 0.2622 | 0.0253 | 0.0159 |
| | | | | | | 26 | 0.9747 | 0.9841 | 0.9738 | 0.1049 | 0.1430 | 0.2479 | 0.0253 | 0.0159 |

## Figures:



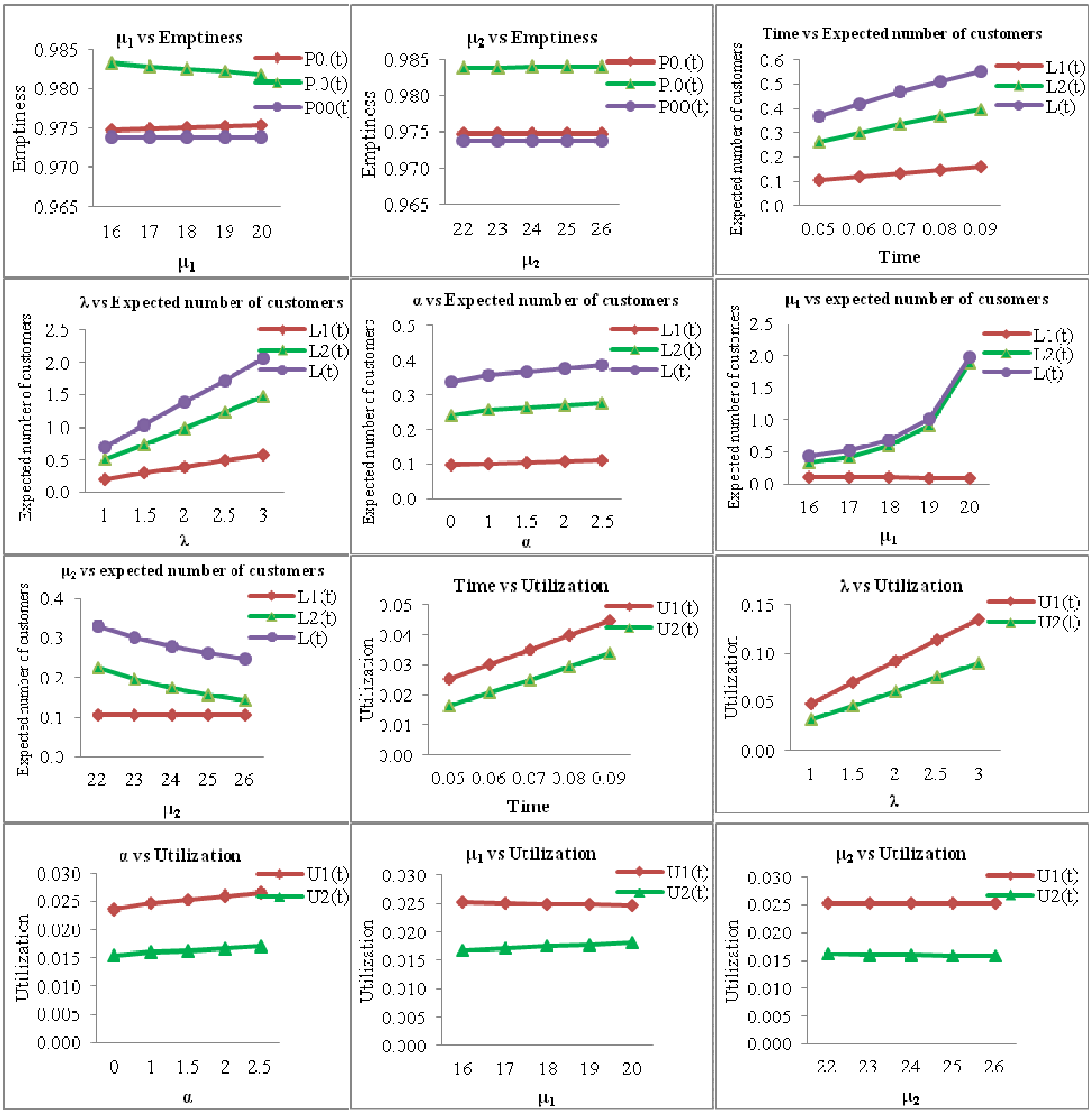



**Figure 2:** The relationships between the parameters and performance measures.

The throughput of the service stations, the average waiting time, the variance, and the coefficient of variation of number of customers in each queue are computed for different values of t, a, b, λ, α, $\mu_1$, $\mu_2$ and presented in Table 2. The relationships between parameters and the performance measures are shown in figure 3.

From Table 2 it can be observed that the throughput of the service stations, waiting time of a customer in each queue, variance and coefficient of variation of number of customers in each queue are highly sensitive with respect to time. As time (t) increases from 0.05 to 0.09, the throughput of the service station, and the variance of the number of customers in each queue increases; the average waiting time of a customer, and the coefficient of variation of number of customers in each queue decreases, when all other parameters are fixed. As the parameter a increases from 2 to 6, the throughput of the service station, the waiting time of a customer in each queue, and the variance of the number of customers in each queue increases, but the coefficient of variation of the number of customer in each queue decreases. The same patterns hold with respect to the parameter b.

**Table – 2:** Values of $Th_1(t)$, $Th_2(t)$, $W_1(t)$, $W_2(t)$, $V_1(t)$, $V_2(t)$, $CV_1(t)$, $CV_2(t)$ for different values of parameter

| t | a | b | λ | α | $\mu_1$ | $\mu_2$ | $Th_1(t)$ | $Th_2(t)$ | $W_1(t)$ | $W_2(t)$ | $V_1(t)$ | $V_2(t)$ | $CV_1(t)$ | $CV_2(t)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 1 | 10 | 0.5 | 1.5 | 15 | 21 | 0.3791 | 0.3430 | 0.2767 | 0.7644 | 0.5719 | 0.3074 | 720.9158 | 211.4556 |
| 0.06 | | | | | | | 0.4540 | 0.4343 | 0.2644 | 0.6910 | 0.6322 | 0.3624 | 662.5917 | 200.5987 |
| 0.07 | | | | | | | 0.5278 | 0.5272 | 0.2536 | 0.6347 | 0.6837 | 0.4141 | 617.9832 | 192.3210 |
| 0.08 | | | | | | | 0.6001 | 0.6209 | 0.2441 | 0.5898 | 0.7286 | 0.4625 | 582.6489 | 185.7110 |
| 0.09 | | | | | | | 0.6706 | 0.7150 | 0.2358 | 0.5529 | 0.7684 | 0.5073 | 554.4493 | 180.1340 |
| | 2 | | | | | | 0.3896 | 0.3680 | 0.2937 | 0.7773 | 0.6334 | 0.3362 | 695.6856 | 202.6660 |
| | 3 | | | | | | 0.3939 | 0.3883 | 0.3147 | 0.7980 | 0.7042 | 0.3660 | 676.7470 | 195.2176 |
| | 4 | | | | | | 0.3958 | 0.4050 | 0.3373 | 0.8239 | 0.7845 | 0.3967 | 663.4607 | 188.7448 |
| | 5 | | | | | | 0.3967 | 0.4189 | 0.3605 | 0.8537 | 0.8743 | 0.4282 | 653.8742 | 182.9894 |
| | 6 | | | | | | 0.3972 | 0.4304 | 0.3841 | 0.8861 | 0.9734 | 0.4607 | 646.5337 | 177.9625 |
| | | 12 | | | | | 0.3822 | 0.3652 | 0.3244 | 0.8485 | 0.7986 | 0.3751 | 720.9922 | 197.6296 |
| | | 14 | | | | | 0.3844 | 0.3831 | 0.3721 | 0.9334 | 1.0630 | 0.4465 | 720.8400 | 186.8587 |
| | | 16 | | | | | 0.3861 | 0.3977 | 0.4199 | 1.0190 | 1.3651 | 0.5215 | 720.7742 | 178.2205 |
| | | 18 | | | | | 0.3874 | 0.4099 | 0.4677 | 1.1050 | 1.7050 | 0.6002 | 720.6808 | 171.0589 |
| | | 20 | | | | | 0.3884 | 0.4201 | 0.5155 | 1.1915 | 2.0826 | 0.6825 | 720.6162 | 165.0291 |
| | | | 1 | | | | 0.7226 | 0.6605 | 0.2790 | 0.7631 | 1.0959 | 0.5922 | 519.2720 | 152.6876 |
| | | | 1.5 | | | | 1.0581 | 0.9731 | 0.2820 | 0.7665 | 1.6200 | 0.8771 | 426.5389 | 125.5579 |
| | | | 2 | | | | 1.3857 | 1.2809 | 0.2851 | 0.7711 | 2.1440 | 1.1619 | 370.6000 | 109.1338 |
| | | | 2.5 | | | | 1.7056 | 1.5840 | 0.2884 | 0.7762 | 2.6680 | 1.4468 | 332.1272 | 97.8308 |
| | | | 3 | | | | 2.0179 | 1.8824 | 0.2917 | 0.7816 | 3.1920 | 1.7317 | 303.5367 | 89.4346 |
| | | | | 0 | | | 0.3525 | 0.3228 | 0.2745 | 0.7493 | 0.5240 | 0.2849 | 748.5816 | 220.7445 |
| | | | | 1 | | | 0.3702 | 0.3363 | 0.2760 | 0.7595 | 0.5560 | 0.2999 | 729.6028 | 214.4210 |
| | | | | 1.5 | | | 0.3791 | 0.3430 | 0.2767 | 0.7644 | 0.5719 | 0.3074 | 720.9158 | 211.4556 |
| | | | | 2 | | | 0.3879 | 0.3498 | 0.2774 | 0.7691 | 0.5879 | 0.3149 | 712.5896 | 208.6095 |
| | | | | 2.5 | | | 0.3968 | 0.3565 | 0.2781 | 0.7735 | 0.6039 | 0.3224 | 704.5422 | 205.8748 |
| | | | | | 16 | | 0.4026 | 0.3515 | 0.2550 | 0.9348 | 0.5530 | 0.3782 | 724.0893 | 187.1515 |
| | | | | | 17 | | 0.4259 | 0.3593 | 0.2361 | 1.1893 | 0.5350 | 0.4815 | 727.7980 | 162.3922 |
| | | | | | 18 | | 0.4488 | 0.3665 | 0.2194 | 1.6122 | 0.5179 | 0.6495 | 730.6119 | 136.4109 |
| | | | | | 19 | | 0.4715 | 0.3731 | 0.2046 | 2.4560 | 0.5017 | 0.9796 | 733.9978 | 108.0157 |
| | | | | | 20 | | 0.4939 | 0.3816 | 0.1914 | 4.9535 | 0.4863 | 1.9579 | 737.9388 | 74.0305 |
| | | | | | | 22 | 0.3791 | 0.3572 | 0.2767 | 0.6291 | 0.5719 | 0.2685 | 720.9158 | 230.5026 |
| | | | | | | 23 | 0.3791 | 0.3713 | 0.2767 | 0.5297 | 0.5719 | 0.2390 | 720.9158 | 248.5390 |
| | | | | | | 24 | 0.3791 | 0.3851 | 0.2767 | 0.4540 | 0.5719 | 0.2159 | 720.9158 | 265.8183 |
| | | | | | | 25 | 0.3791 | 0.3987 | 0.2767 | 0.3946 | 0.5719 | 0.1971 | 720.9158 | 282.2374 |
| | | | | | | 26 | 0.3791 | 0.4121 | 0.2767 | 0.3470 | 0.5719 | 0.1816 | 720.9158 | 298.0039 |

It can be further observed that as the parameter (λ) increases from 1 to 3, the throughput of the service station, the waiting time of the customer in each queue, and the variance of the number of customers in each queue increases, whereas the coefficient of variation of the number of customers in each queue decreases. The same patterns hold with respect to the parameter (α).

It can also be observed that as the service rate ($\mu_1$) increases from 16 to 20, the throughput of the service stations increases; the average waiting time, and the variance of the number of customers in the first queue decreases, but the second queues increases; and the coefficient of variation of the number of customers in the first queue increases, but the second queue decreases. It can also be observed that as the service rate ($\mu_2$) increases from 22 to 26, the throughput of the second service station increases, but the first service station remains constant; the average waiting time, and the variance of the number of customers in the second queue decreases, but the first queue remains constant; and the coefficient of variation of the number of customers in the second queue increases, but the first queue remains constant.

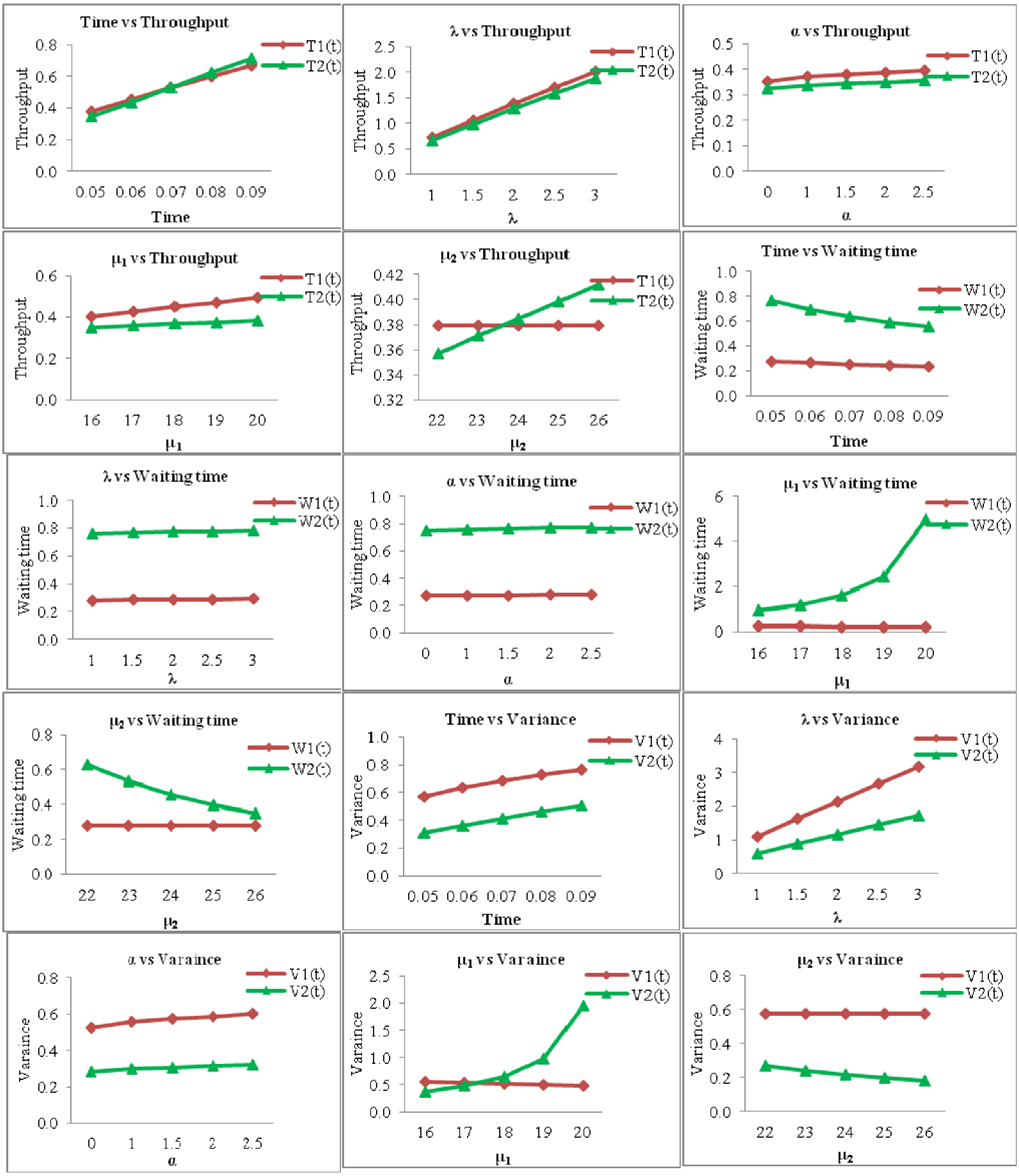



**Figure 3:** The relationships between the parameters and performance measures.

From this analysis it can also be observed that the bulk size distribution parameters and composite mean arrival significantly influences the system performance measures.

## 6. COMPARATIVE STUDY

A comparative study of the developed model with that of homogeneous compound Poisson arrivals is carried by taking $\alpha = 0$ in the model and different values of t. Table 3 shows the points study of having models with homogeneous and non-homogeneous compound Poisson arrivals.

**Table 3:** Comparative study of models with non- homogeneous and homogeneous Poisson arrivals

| t | Parameters Measured | $\alpha = 1$ | $\alpha = 0$ | Difference | Percentage of Variation |
|---|---|---|---|---|---|
| 0.05 | $L_1(t)$ | 0.1022 | 0.0967 | 0.0055 | 5.6877 |
| | $L_2(t)$ | 0.2554 | 0.2418 | 0.0136 | 5.6245 |
| | $U_1(t)$ | 0.0247 | 0.0235 | 0.0012 | 5.1064 |
| | $U_2(t)$ | 0.0160 | 0.0154 | 0.0006 | 3.8961 |
| | $T_1(t)$ | 0.3702 | 0.3525 | 0.0177 | 5.0213 |
| | $T_2(t)$ | 0.3363 | 0.3228 | 0.0135 | 4.1822 |
| | $W_1(t)$ | 0.2760 | 0.2745 | 0.0015 | 0.5464 |
| | $W_2(t)$ | 0.7595 | 0.7493 | 0.0102 | 1.3613 |
| 0.06 | $L_1(t)$ | 0.1163 | 0.1088 | 0.0075 | 6.8934 |
| | $L_2(t)$ | 0.2907 | 0.2720 | 0.0187 | 6.8750 |
| | $U_1(t)$ | 0.0294 | 0.0277 | 0.0017 | 6.1372 |
| | $U_2(t)$ | 0.0202 | 0.0192 | 0.0010 | 5.2083 |
| | $T_1(t)$ | 0.4414 | 0.4162 | 0.0252 | 6.0548 |
| | $T_2(t)$ | 0.4239 | 0.4033 | 0.0206 | 5.1079 |
| | $W_1(t)$ | 0.2635 | 0.2614 | 0.0021 | 0.8034 |
| | $W_2(t)$ | 0.6858 | 0.6744 | 0.0114 | 1.6904 |
| 0.07 | $L_1(t)$ | 0.1290 | 0.1192 | 0.0098 | 8.2215 |
| | $L_2(t)$ | 0.3224 | 0.2979 | 0.0245 | 8.2242 |
| | $U_1(t)$ | 0.0341 | 0.0318 | 0.0023 | 7.2327 |
| | $U_2(t)$ | 0.0244 | 0.0230 | 0.0014 | 6.0870 |
| | $Th_1(t)$ | 0.5109 | 0.4770 | 0.0339 | 7.1069 |
| | $Th_2(t)$ | 0.5125 | 0.4832 | 0.0293 | 6.0637 |
| | $W_1(t)$ | 0.2524 | 0.2498 | 0.0026 | 1.0408 |
| | $W_2(t)$ | 0.6290 | 0.6166 | 0.0124 | 2.0110 |
| 0.08 | $L_1(t)$ | 0.1404 | 0.1281 | 0.0123 | 9.6019 |
| | $L_2(t)$ | 0.3509 | 0.3203 | 0.0306 | 9.5535 |
| | $U_1(t)$ | 0.0386 | 0.0356 | 0.0030 | 8.4270 |
| | $U_2(t)$ | 0.0286 | 0.0267 | 0.0019 | 7.1161 |
| | $T_1(t)$ | 0.5784 | 0.5347 | 0.0437 | 8.1728 |
| | $T_2(t)$ | 0.6012 | 0.5617 | 0.0395 | 7.0322 |
| | $W_1(t)$ | 0.2427 | 0.2396 | 0.0031 | 1.2938 |
| | $W_2(t)$ | 0.5837 | 0.5702 | 0.0135 | 2.3676 |
| 0.09 | $L_1(t)$ | 0.1507 | 0.1358 | 0.0149 | 10.9720 |
| | $L_2(t)$ | 0.3767 | 0.3395 | 0.0372 | 10.9573 |
| | $U_1(t)$ | 0.0429 | 0.0393 | 0.0036 | 9.1603 |
| | $U_2(t)$ | 0.0328 | 0.0304 | 0.0024 | 7.8947 |
| | $T_1(t)$ | 0.6434 | 0.5890 | 0.0544 | 9.2360 |
| | $T_2(t)$ | 0.6895 | 0.6384 | 0.0511 | 8.0044 |
| | $W_1(t)$ | 0.2342 | 0.2306 | 0.0036 | 1.5611 |
| | $W_2(t)$ | 0.5464 | 0.5318 | 0.0146 | 2.7454 |

From the table 3 it can also be observed that as time increases, the percentage variation of the performance measures between the models also increases. The model with non-homogeneous compound Poisson arrivals has higher utilisation than the model with homogeneous compound Poisson arrivals. It can also be observed that the assumption of non-homogenous compound Poisson arrivals has a significant influence on all the performance measures of the queueing model. Time also has a significant effect on the system performance measures, and this model can predict the performance measures more accurately. This model also includes some of the earlier models as particular cases.

## 7. CONCLUSION:

This paper addresses the development and analysis of a two-node tandem queueing model with non-homogeneous bulk arrivals and a state-dependent service rate. It is also assumed that the two queues are connected in tandem and after getting service from the first queue the customer join the second queue. It is also assumed that the customers arrive in bulk groups of random size depending on time. The arrival process of the queue is characterised by a non-homogeneous compound Poisson process. The explicit expressions for the system characteristics, such as the average number of customers in each queue, the probability that the queue is empty, the average waiting time of a customer in the queue, and the throughput of the service station are derived explicitly. The sensitivity of the model revealed that the bulk size distribution parameters have a significant influence on the system performance measures. By regulating the bulk size distribution parameters the congestion in queues and the mean delay can be reduced. A comparative study of the developed model with a model using homogeneous compound Poisson arrivals revealed that time has a significant effect on system performance measures and the performance measures can be predicted more accurately and realistically using the developed model. This model also includes several of the earlier existing models as particular cases for specific values of the parameters. This model can also be extended by obtaining the optimal values of the model parameters under cost considerations, which will be pursued elsewhere.

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## REFERENCES:

1. Abryen P, , Baraniuk R, Fland P, Riedi R, Veitch D, (2002) Multiscale nature of network traffic, *IEEE Signal Processing Magazine*, Vol.19, Issue.3, pp 28-46.
2. Ahmed, M.M. S (2007), Multi- Channel bi-level heterogeneous servers bulk arrivals queueing system with Erlangian service time. *Mathematical and Computational Applications*, Vol 12, No.2, pp 97-105.
3. Brockmeyer, E., Halstrom, H.L. and Jensen, A. (1948), The life and works of A.K. Erlang, *The Copenhagen Telephone Company, Copenhagen, Denmark.*
4. Cappe O, Moulines E, Pesquet J C, Petropulu and Y and X (2002), Long Range Dependence and Heavy- trail Modelling for Tele traffic Data, *IEEE Signal Processing Magazine*, pp 14-27.
5. Charan Jeet Singh, Madhu Jain and Binay Kumar (2011), Queueing model with state dependent bulk arrival and second optional service, *International Journal of Mathematics in Operational Research,* Vol3, No.3, pp 322-340.
6. Chaudhry, M.L and Chang, S.H. (2004), Analysis of the discrete time bulk service queue Geo/$G^Y$/1/N+B. *Operations Research Letters*, Vol.32, Issue 4,pp 355-363.
7. Che Soong Kim, Seog Ha Park, Alexander Dudin, Valentina Klimenokand Gennady Tsarenkov (2010), Investigation of the BMAP/G/1→/PH/1/M Tandem queue with retrials and losses, *Applied Mathematical Modelling*, Vol. 34, No. 10, pp 2926-2940.
8. Che Soong Kim, Valentina Klimenok and Olga Taramin (2010), A tandem retrial queueing system with two markovian flows and reservation of channels, *Computers and Operations Research*, Vol.37, No.7, pp 1238-1246.
9. Chen A., Pollett P., Li J., and Zhang, H. (2010), Markovian Bulk arrival and Bulk –Service Queues with State Dependent Control, *Queueing Systems: Theory and Applications,* Vol .64, Issue 3, pp 267-304.
10. Choi, B.D. and Choi, D.I (1996), Queueing system with length dependence service times and its applications to cell discarding scheme in ATM networks, *IEEE Proceedings Communications*, Vol. 143, pp 5-11.

11. Crovella ME and Bestarros A (1997), Self Similarly in worldwide traffic: evidence and possible causes, *IEEE/ACM Transactions on Networking*, Vol.5, No.6, pp 835-846.
12. Dieter Claeys, Joris Walraevens, Koenraad Laevens and Herwig Bruneel (2010), Analysis of threshold-batch service queueing system with batch arrivals and general service times, *European Journal of Operations Research,* Vol.207, No.2, pp 827-835.
13. Dinda P.A., (2006), " Design, Implementation and performance of an Extensible toolkit in resource prediction in Distributed systems", *IEEE Transactions on Parallel and Distributed systems*, Vol. 17, No.2, pp 160-173.
14. Erlang, A.K.(1909). Probability and telephone calls, *Nyt. Tidsskr Krarup Mat. Ser. B.* Vol. 20, pp. 33-39.
15. Feldmann A, (2000), Characteristics of TCP connection arrivals, Chapter 15, self similar network traffic and performance evaluation, edited by *K. Park, W. Willinger*, John Wiley & Sons Inc.
16. Fischer M J, Gross D, Masi D and Shortle J F (2001), Analyzing the waiting time process is internet queueing systems with the transform approximation method, *The telecommunication review*,12, pp 21-32.
17. Haridass, M. And Arumuganathan, R (2011), Analysis of a batch arrival general bulk service queueing system with variant threshold policy for secondary jobs, *International Journal of Mathematics in Operational Research,* Vol.3, No.1, pp 56-77.
18. Jackson R.R.P (1956), Random Queueing Process with Phase type service, *Journal of Royal Statistical Society, Series B*, Vol. 18, No. 1, pp 129-132.
19. Jackson, R.R.P (1954), Queueing systems with Phase type service, *Operations Research Quarterly*, Vol.5, No. 4, pp 109-120.
20. Juan, M.T. (2005). Numerical methods for the single-server bulk-service queueing system with variable service capacity, $M/G^Y/1$, with discretized service time probability distribution. *International Conference on Operations Research*, Sept. 7-9, Berman, Germany.
21. K. Srinivasa Rao, Prasad Reddy and P. Suresh Varma (2006), Interdependent Communication Network with Bulk arrivals, *International Journal of Management and Systems*, Vol. 22, No. 3, pp 221-234.
22. Kin K.Leung (2002), Load –dependent service queues with application to congestion control in broadband networks, *Performance Evaluation*, Vol.50, No.4, pp 27-40.
23. Leland et al (1994), On the self similar nature of Ethernet traffic (Extended version), *IEEE/ACM Transactions on Networking*, Vol.2, No.1, pp 1-15.
24. Lieshout, P and Mandjes, M. (2006), Tandem Brownian queues, *Journal of Probability, Networks and Algorithms*, Report PNA- RO604, Vol.4, pp 1-20.
25. Madan, K.C., Al- Naseer, A.D., and Al-Masri (2004). On $M^{[X]}/\binom{G_1}{G_2}/1$ queue with optional re-service, *Applied Mathematics and Computation*, Vol. 152, A.Q., pp. 71-88.
26. Murali Krishna P, Gadre VM, Desai U B (2003), Multi fractal based network traffic modelling, Kluwer academic publishers.
27. Nageswararao, K., Srinivasarao, K and Srinivasarao, P (2010), A tandem communication network with dynamic bandwidth allocation and modified phase type transmission having bulk arrivals, *International Journal of Computer Science*, Issues, Vol. 7, No.2, pp 18-26.
28. O. Brien, G.G (1954), The Solution of Some Queueing Problems, *Journal of the Society for Industrial and Applied Mathematics*, Vol.2, pp 132-142.
29. Padmavathi, G., Srinivasarao, K and Reddy, K.V.V.S (2009), Performance Evaluation of Parallel and Series Communication network with dynamic bandwidth allocation CIIT *International Journal of Networking and Communication*, Vol.1, No.7, pp 410-421.
30. Parthasarathy, P.R and Selvarju, N. (2001), Transient analysis of a queue where potential customers are discouraged by queue length, *Mathematical problems in Engineering*, Vol.7, pp 433-454.
31. Paul J. Burke, (1956), The output of a queueing system, *Journal of Operations Research*, Vol.4, No.6, pp 699-704.
32. Rakesh Singhai, Shiv Dutt Joshi and Rajendra K P Bhatt (2007), A Novel Discrete distribution and process to model self similar traffic, *9th IEEE International Conference on Telecommunication* – ConTel, pp 167-172.
33. Schleyer, M. and Furmans, K. (2007). An analytical method for the calculation of the waiting time distribution of a discrete time G/G/1-queueing systems with batch arrivals. *OR Spectrum*, Vol. 29, No. 4, pp. 745-763.
34. Srinivasa Rao, K., Vasanta, M.R., and Vijaya Kumar, C.V.R.S., (2000), On an interdependent communication Network, *Opsearch* Vol. 37, No.2, pp 134-143.
35. Suresh Varma, P. and Srinivasa Rao, K. (2007), A Communication Network with load dependent transmission, *International Journal of Mathematical Science*, Vol. 7, No.2, pp 199-210.
36. Wai Kin Victor Chan and Lee W. Schruben (2008), Mathematical Programming models of closed tandem queueing networks, *Journal of ACM Transactions on Modelling and Computer Simulation*, Vol.19, No.1.
37. Yongjian Li, Xiaoqiang cai, Fengsheng Tu and Xiulishao (2004), Optimization of Tandem queue systems with finite buffers, *Computers and Operations Research*, Vol.31, pp 963-984.
38. Yukuo Hayashida (1993), Throughput analysis of tandem type go-back-NARQ Scheme for Satellite Communications, *IEEE Transaction on Communications*, Vol.41, No. 10, pp 1517-1524.