International Journal of Advanced Computer and Mathematical Sciences
ISSN 2230-9624. Vol 3, Issue 4, 2012, pp 510-519
http://bipublication.com

THE POPULATION MODEL OF LAGOS STATE, NIGERIA AND CHAOS THEORY.

Ometan O. O., Olusa T. V., Adegbola R. B. Oseni S. O., Atutuomah E S.

Department of Physics, Lagos State
University, Ojo, Lagos, Nigeria (ometanfunmi@gmail.com)

[Received-21/03/2012, Accepted-01/10/2012]

ABSTRACT

This research work is mainly to showing the various behavior of a population based on the Lagos State population census data. The carrying capacity of the state was determined using the logistic equation. And the various behavior of the state population over time was shown using a time series plot and the logistic map. The result shows period doubling bifurcation and chaotic behavior which is identical in pattern to those from the simple mathematical models of logistic growth in ecology. The result from this study helps to understand chaotic dynamics, logistic growth equation and population dynamics.

Keywords: logistics equation, Chaos, Bifurcation, growth rates, population dynamics and carrying capacity.

INTRODUCTION

A population is "a collection of plants, animals, or other organisms, all of the same species, that live together and reproduce" [1]. Population dynamics is the study of marginal and long term changes in the number of individuals, sex, and weight and age composition in a particular location. Since population is a function of time, it has been seen to exhibit some chaotic behaviour at some point in time.

Chaos is the term used to describe the apparently complex behaviour of what we consider to be a simple well – behaved system. The key element in understanding chaotic system is the understanding of non-linearity. The theory of nonlinear dynamics allows us to describe and classify the complex behaviour of these kinds of system, and theory of chaos allows us to see an order and universality that underlies these complexities

The behavior of population is also seen to be chaotic, which implies that population growth rate exceeds the rate at which density-dependence feeds back into the process, that is, the population overcompensates for the existing density by either growing too fast or declining too fast and overshooting Carrying Capacity *K*.

Stability is an important concept when studying systems over a given time interval. A system is considered stable when a condition converges toward a single point within a set range. On the other hand, a system becomes unstable when conditions diverge from a fixed point and depart from this range. Further, when the system diverges and splits, it creates a more complicated system. The locations of these splits are called bifurcation points and as the system progresses with time it exponentially develops more bifurcation points.

The concept of chaos in ecological populations is widely known for non-overlapping generations since the early theoretical works of May [1974, 1976], and successively applied to laboratory and field studies [2]. A classical approach using very simple models consist of using discrete, first-order non-linear difference equations for populations with N_i individuals at time i of the form $N_{t+1} = f(N_t)$, where $f(N_t) = aN_t g(N_t,\ldots,N_{t-j})$ and $g(N_t,\ldots,N_{t-j})$ is some nonlinear function describing some degree of density-dependence with time delay j. In fact, a well-known equation describing a full range of dynamic behaviour was developed by Pierre Verhulst: $N_{t+1} = N_t\left[1 + R_0\left(1 - \frac{N_t}{K}\right)\right]$, where R_0 stands for the discrete initial growth rate, and N_t the initial population. Keyfitz and Flieyer [3] were the first to analysis the human population in their work on world population growth and aging. An illustration of a population which is growing exponentially has results described in Rubinow [4].

Ibrahim and Lewis [5] used the logistics model to study and determine the population growth and projection of the people in Gwer local government area of Benue State. Ogbeide and Ikpotokin [6] used the logistic model to predict if by the year 2020, the population of Esan Local Government in Edo State Nigeria will be chaotic or not. All these led to this research to examine the status of the Lagos State population.

METHODOLOGY

The following assumptions were applied to this study :

i. Age and sex differences between the population can be ignored

ii. Each member of the population has an equal chance of dying and surviving.

iii. The population is isolated, that is no immigration or emigration or that immigration equal to emigration.

iv. Birth rate and death rate are proportional to the size of the population at any given time.

v. The rate of growth of the population is proportional to the size of the population.

vi. Population census data are taken at the end of the year.

The Logistic Equation

The simplest model was Malthus model, the model accounts for continuous exponential growth of a population. It is given by

$$N_t = N_0 e^{rt} \qquad 1.0$$

where N(t) is the variable that indicates the value of the population, a function of time, N_o is the initial value of the population and r is the constant growth of the population.

$$N_t = \lambda N_o$$

Where $e^{rt} = \lambda$. 1.1

If $\lambda > 1$, then the population increases. If $\lambda < 1$, the population decreases. If $\lambda > 1$ and remains constant for subsequent generations, the population would continue to increase, leading to Malthusian population explosion.

Differentiating equation 1.0, we have

$$\dot{N}_{(t)} = rN_{(t)} \qquad 1.2$$

this, as is common practice, can be put in discrete form

$$N_{(t+1)} = N_{(t)} + rN_{(t)} \qquad 1.3$$

In the above case, $N_{(t)}$ indicates the population at the beginning of the time interval of amplitude $\Delta t = 1$, a population that remains constant for the entire duration of the interval. A differential equation that describes a dynamical system in continuous time is called a flow, while the version in discrete time, such as the equation 1.3, is known as a map.

Verhulst introduced the carrying capacity; a constant indicated by K. When the number N of individuals in the population reaches the Value K, growth itself becomes nil. However, this is not enough, K must display it influence even before N reaches the maximum permitted amount. It therefore has to show it presence by curbing the population's speed of growth to an extent that is proportional to the size of N

Thus Verhulst obtained by summing an approximately selected negative term that is a function of the value of population N to the second term of the equation A and to it discrete version.

In this way, he introduced a real feedback to the system; population growth is now controlled.

According to Verhulst assumption, one of many ones, Malthus model in discrete form can be adapted with the introduction of the negative corrective term, proportional to the product of growth rate and the square of the value of population , follows

$N_{(t+1)} = N_{(t)} + R_o N_{(t)}\left(1 - \frac{N_{(t)}}{K}\right)$

$N_{(t+1)} = N_{(t)}\left[1 + R_0\left(1 - \frac{N_{(t)}}{K}\right)\right]$ 1.4

This equation can be usefully re –written in a more concise and schematic form, that itself better to subsequent analyses

$$X_{(t+1)} = R_o X_{(t)}[1 - X_{(t)}] \qquad 1.5$$

To analytically examine the maximum growth rate of the population, we recall the logistic equation 1.4

$$N_{(t+1)} = N_{(t)}\left[1 + R_0\left(1 - \frac{N_{(t)}}{K}\right)\right]$$

Expanding

$$N_{(t+1)} = N_t + N_t R_o\left(1 - \frac{N_t}{K}\right)$$

$$N_{(t+1)} - N_t = N_t R_o\left(1 - \frac{N_t}{K}\right)$$

$$N_{(t+1)} - N_t = N_t R_o - \left(\frac{R_o}{K}\right)N_t^2$$

To obtain maximum population growth with respect to N_t, we take the derivative and set to zero. Thus

$$\frac{\partial(N_{(t+1)} - N_t)}{\partial N_t} = R_o - \frac{2R_o N_t}{K}$$

Setting the result to zero gives the value of N_t that maximizes the annual increase in the population. i.e.

$$R_o - \frac{2R_o N_t}{K} = 0$$

$$=> R_0\left(1 - \frac{2N_t}{K}\right) = 0$$

Now we have two terms multiplied by each other. For the equation to be zero, one must be zero, so either R_o is zero or $\left(1 - \frac{2N_t}{K}\right)$ is zero. If R_o were

zero, it would mean that the maximum possible growth rate for a population is zero. Such a population would be dead before it could get started so this situation is not ecologically relevant. We'll just consider the case for which $\left(1 - \frac{2N_t}{K}\right)$ is zero.

Set $\left(1 - \frac{2N_t}{K}\right)$ equal to zero and solve for N_t;

$$\left(1 - \frac{2N_t}{K}\right) = 0$$

$$1 = \frac{2N_t}{K}$$

$$K = 2N_t$$

$$N_t = \frac{K}{2} \qquad 1.6$$

So the maximum population growth rate is at half the carrying capacity.

DATA USED

Lagos state has approximately 17,552,942 inhabitants in the Census Report [7].

The projected annual growth rate from 1991 population census was 7.5%, where the total population was estimated to be 5725116 people.

Census Data obtained from federal bureau of statistics and from National population census commission for Lagos state, Nigeria

Table 1.1: Year of census and total population

Year of census	Total population
1911	73,766
1921	99,690
1931	126,108
1952	272,000
1963	665,000
1991	5,725,116
2006	17,552,942

CALCULATED DATA

To determine the carrying capacity (K), refer to logistic equation and census data. The estimate comes from the Malthusian law applied to the growth between years of data collected. From equation 1.0

$$N_{(t)} = N_o e^{R_o(t - t_o)},$$

from which we get

$$R_o = \frac{In\frac{N_{(t)}}{N_{(o)}}}{(t - t_o)} \qquad 1.12$$

From equation 1.12, the corresponding values for R_o for various years are given in the table 1.2.

Table 1.2 : A tables showing the growth rate over time in Lagos state

YEAR	GROWTH RATE (Ro)
1911 and 1921	0.03012
1921 and 1931	0.02351
1931 and 1952	0.0366
1952 and 1963	0.08127
1963 and 1991	0.07689
1991 and 2006	0.07469

The remaining parameter to determine is the maximum sustainable population K. Basically we will a value of K, plot the theoretical curve and the actual data on a single graph. Then vary K until we get a match, if possible. We define a routine now called logistic graph which when executed will construct the theoretical graph with a supplied value of K.

RESULTS

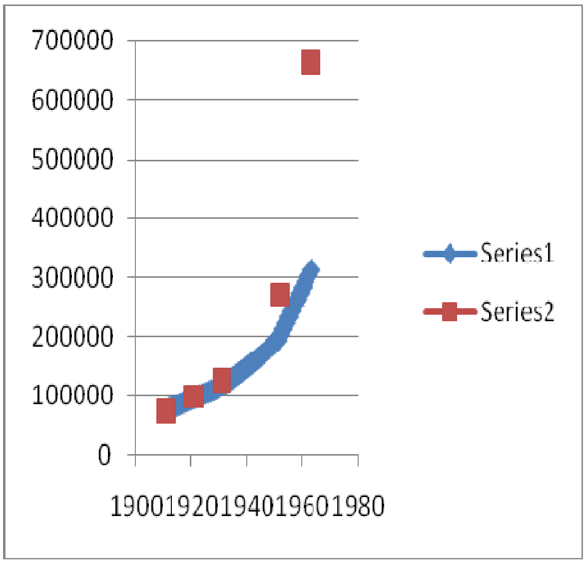

Fig. 1.1 A graph of population against time at K = 0.5 million

We see that the fit is not bad, but the theoretical curve lies below the data. On trying larger values of K it was discovered that the best fit for both plots occur at K equals three million

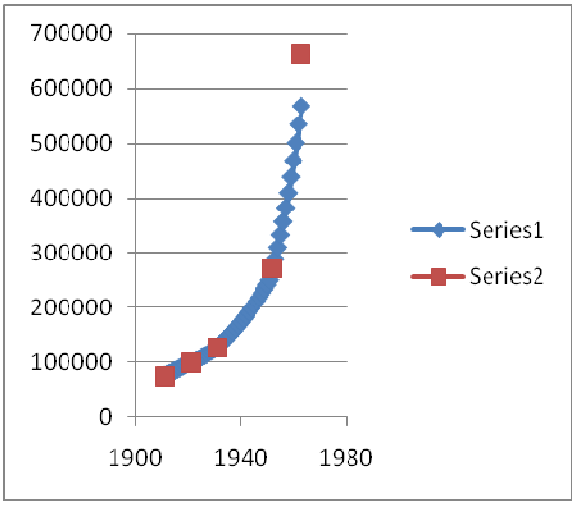

Fig. 1.2: A graph of population against time at K = 3 million (from table 1.3)

Where series 2 is a plot of census data obtained, and series 1 is a plotted of data obtained by the use of logistic equation.

From graphs above (fig 1.1 & 1.2), the curve having the best fit at the early period of population growth before the exponential increase in population was that for which carrying capacity K is 3 million.

Therefore, the carrying capacity of the state is 3 million

TABLE 1.3 A table showing logistic equation (equation 1.4) for which K = 3 million

carrying cap	Year	Nt	R0	Lamda [Ro (1-Nt/K)]	Nt+1
3000000	1911	73766	0.03012	0.029379	75933.2
3000000	1912	75933.2	0.03012	0.029358	**78162.42**
3000000	1913	78162.42	0.03012	0.029335	**80455.33**
3000000	1914	80455.33	0.03012	0.029312	**82813.66**
3000000	1915	82813.66	0.03012	0.029289	**85239.15**
⋮	⋮	⋮	⋮	⋮	⋮
3000000	1919	92936.8	0.03012	0.029187	**95649.34**
3000000	1920	95649.34	0.03012	0.02916	**98438.44**
3000000	1921	98438.44	0.02351	0.022739	**100676.8**
3000000	1922	100676.8	0.02351	0.022721	**102964.3**
⋮	⋮	⋮	⋮	⋮	⋮
3000000	1929	117777.9	0.02351	0.022587	**120438.2**

3000000	1930	120438.2	0.02351	0.022566	**123156**
3000000	1931	123156	0.0366	0.035097	**127478.5**
3000000	1932	127478.5	0.0366	0.035045	**131945.9**
⋮	⋮	⋮	⋮	⋮	⋮
3000000	1951	242612	0.0366	0.03364	**250773.5**
3000000	1952	250773.5	0.08127	0.074477	**269450.2**
3000000	1953	269450.2	0.08127	0.073971	**289381.6**
⋮	⋮	⋮	⋮	⋮	⋮
3000000	1962	501458.1	0.08127	0.067686	**535399.5**
3000000	1963	535399.5	0.07689	0.063168	**569219.5**
3000000	1964	569219.5	0.07689	0.062301	**604682.3**
⋮	⋮	⋮	⋮	⋮	⋮
3000000	1981	1377897	0.07689	0.041575	1435182
3000000	1982	1435182	0.07689	0.040106	1492742
3000000	1983	1492742	0.07689	0.038631	1550408
⋮	⋮	⋮	⋮	⋮	⋮
3000000	1990	1889223	0.07689	0.028469	1943008
3000000	1991	1943008	0.07469	0.026316	1994139
3000000	1992	1994139	0.07469	0.025043	2044078
⋮	⋮	⋮	⋮	⋮	⋮
3000000	2005	2526567	0.07469	0.011787	2556347
3000000	2006	2556347	0.07469	0.011045	2584583
3000000	2007	2584583	0.07469	0.010342	2611314
⋮	⋮	⋮	⋮	⋮	⋮
3000000	2018	2806660	0.07469	0.004814	2820170
3000000	2019	2820170	0.07469	0.004477	2832797
3000000	2020	2832797	0.07469	0.004163	2844589

The carrying capacity of the state is 3 million, so half the carrying capacity will be 1.5 million. Table 1.3, this occur in 1982.

The maximum growth is obtained by calculating the growth rate using the logistic equation and taking the population at 1982 as final population and that of 1911 as initial population.

From Malthusian law in discrete form (equation 1.5)

$$N_{(t+1)} = N_{(t)} + R_o N_{(t)}$$

$$N_{(t+1)} = N_{(t)} (1 + R) \qquad 1.7$$

This implies that $N_t = N_o (1+ R)^t$. 1.8

Since equation 1.7 can be written as equation 1.8, then we can write for logistic equation (i.e. equation 1.4) as

$$N_{(t)} = N_o\left[1 + R_0\left(1 - \frac{N_0}{K}\right)\right]^t \qquad 1.9$$

Making R_o the subject, we have

$$\frac{\left(\frac{N_t}{N_o}\right)^t - 1}{\left(1 - \frac{N_t}{K}\right)} = R_0 \qquad 1.10$$

Inserting values for N_t, N_o and K, we obtain maximum Ro to be R_{om} = 0.08612, where R_{om} is the maximum growth rate.

To obtain the corresponding value of growth in the logistic map (for which r is between 0 and 4), we say

if the value 0.08612 for R_{om} corresponds to the value 4 for r,

Then

$$r = \frac{4R_0}{R_{om}} \qquad 1.11$$

To obtain a table for the logistic map equation, we derive the various corresponding values of r for the different R_o using equation 1.11

For Ro = 0.03012, r = 1.3990
Ro = 0.02351, r = 1.0920
Ro = 0.03660, r = 1.70000
Ro = 0.08127, r = 3.7747
Ro = 0.07689, r = 3.5713
Ro = 0.07469, r = 3.4691.

Inserting these values for r in the logistic map equation gives the logistic map table for the years using equation 1.5.

Table 1.4 Difference equation using various corresponding values of R_o for r

YEAR	Nt	r	rNt(1 - Nt)
1911	0.0245887	1.399	0.0335537
1912	0.0335537	1.399	0.0453666
1913	0.0453666	1.399	0.0605885
⋮	⋮	⋮	⋮
1920	0.2109991	1.399	0.2329034
1921	0.2329034	1.092	0.1950961
1922	0.1950961	1.092	0.1714807
⋮	⋮	⋮	⋮
1930	0.1119584	1.092	0.1085707

1931	0.1085707	1.7	0.1645313
1932	0.1645313	1.7	0.2336833
⋮	⋮	⋮	⋮
1951	0.4117647	1.7	0.4117647
1952	0.4117647	3.7747	0.9142872
1953	0.9142872	3.7747	0.2958086
⋮	⋮	⋮	⋮
1962	0.6299772	3.7747	0.879905
1963	0.879905	3.5713	0.3773872
1964	0.3773872	3.5713	0.8391344
⋮	⋮	⋮	⋮
1990	0.5039156	3.5713	0.8927702
1991	0.8927702	3.4691	0.3321023
1992	0.3321023	3.4691	0.7694823
⋮	⋮	⋮	⋮
2005	0.8659867	3.4691	0.4026021
2006	0.4026021	3.4691	0.8343659
2007	0.8343659	3.4691	0.4794277
⋮	⋮	⋮	⋮
2018	0.4034774	3.4691	0.8349547
2019	0.8349547	3.4691	0.4780605
2020	0.4780605	3.4691	0.8656052

A time series plot of population of Lagos State using table 1.4 is shown below

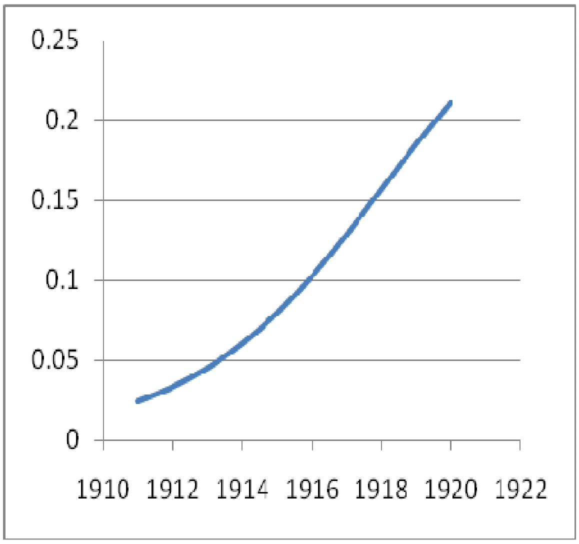

Fig 1.3: A GRAPH OF POPULATION AGAINST TIME BETWEEN 1911 - 1920

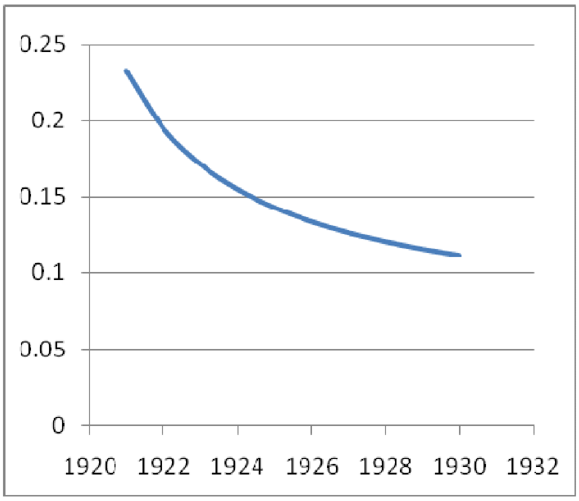

Fig 1.4: A GRAPH OF POPULATION AGAINST TIME BETWEEN 1921 - 1930

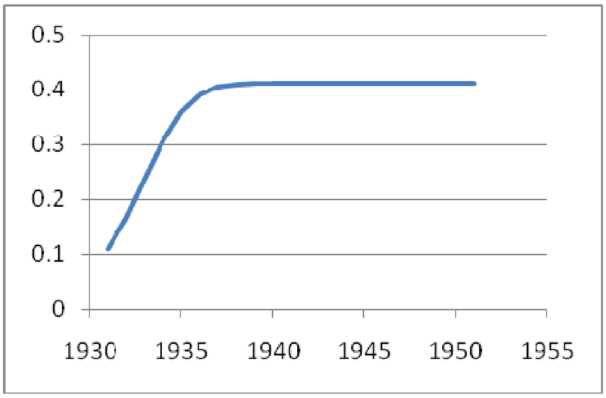

Fig 1.5: A GRAPH OF POPULATION AGAINST TIME BETWEEN 1931 - 1951

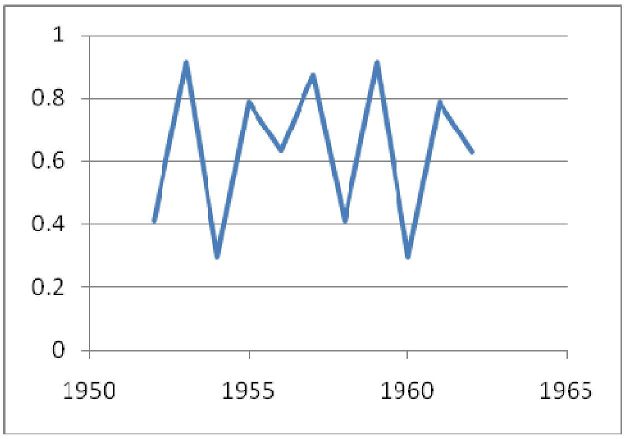

Fig 1.6: A GRAPH OF POPULATION AGAINST TIME BETWEEN 1952 - 1962

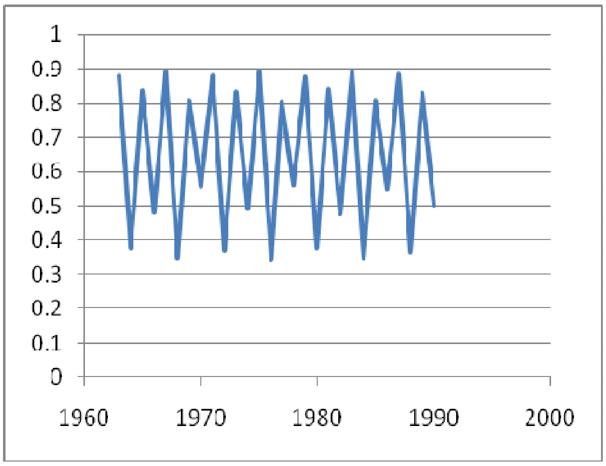

Fig 1.7: A GRAPH OF POPULATION AGAINST TIME BETWEEN 1963 - 1990

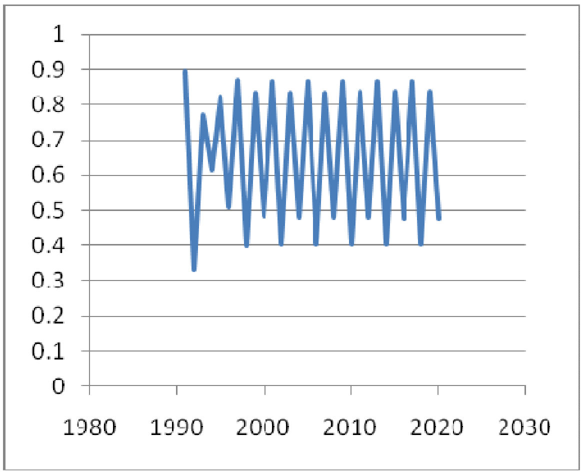

Fig 1.8: A GRAPH OF POPULATION AGAINST TIME BETWEEN 1991 - 2020

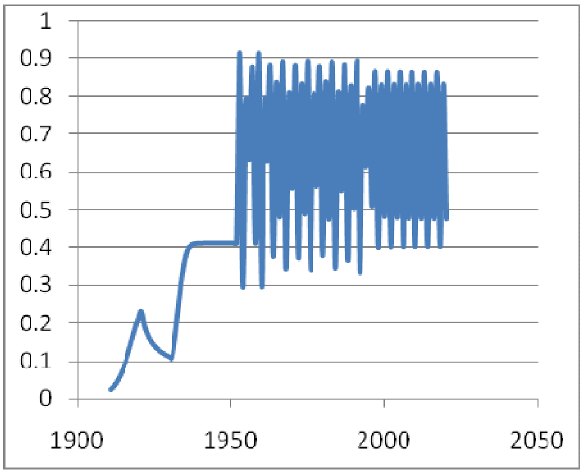

Fig 1.9: A PLOT OF POPULATION AGAINST TIME BETWEEN 1911 - 2020

Plotting the logistic map using table 1.4, which is a map of $N_{(t+1)}$ against N_t we obtain

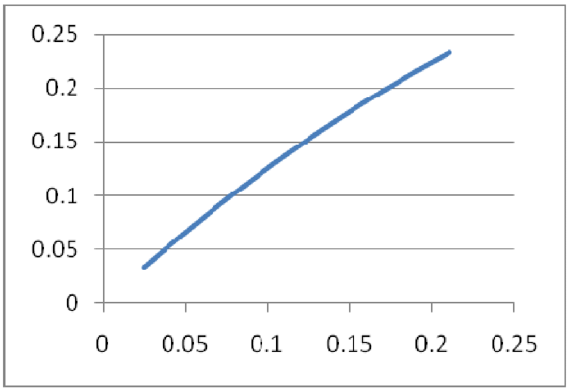

Fig 1.10: LOGISITIC MAP AT r = 1.399 (1911 - 1920)

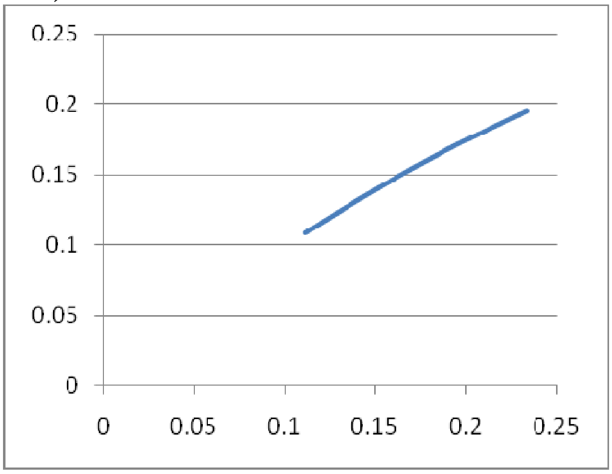

Fig 1.11: LOGISITIC MAP AT r = 1.092 (1921 - 1930)

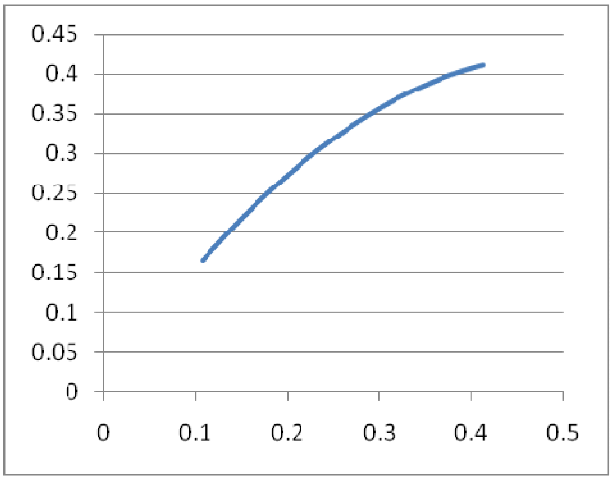

Fig 1.12: LOGISITIC MAP AT r = 1.700 (1931 - 1952)

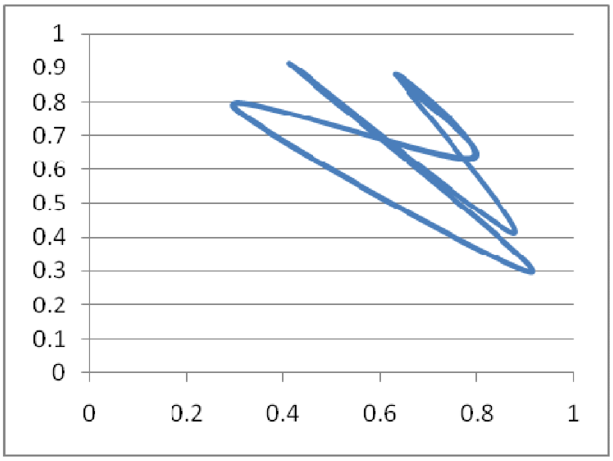

Fig 1.13: LOGISITIC MAP AT r = 3.7747 (1952 - 1962)

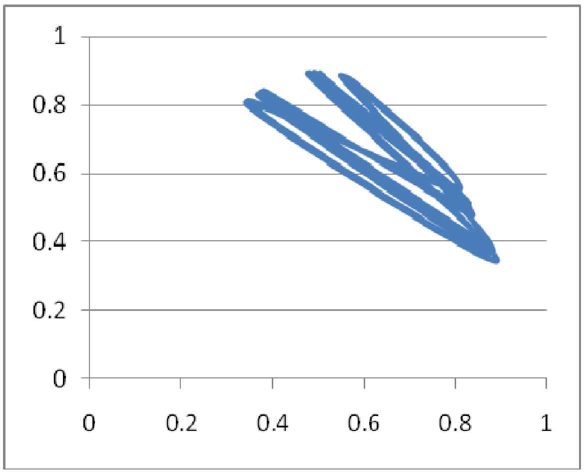

Fig 1.14: LOGISITIC MAP AT r = 3.5714 (1963 - 1990)

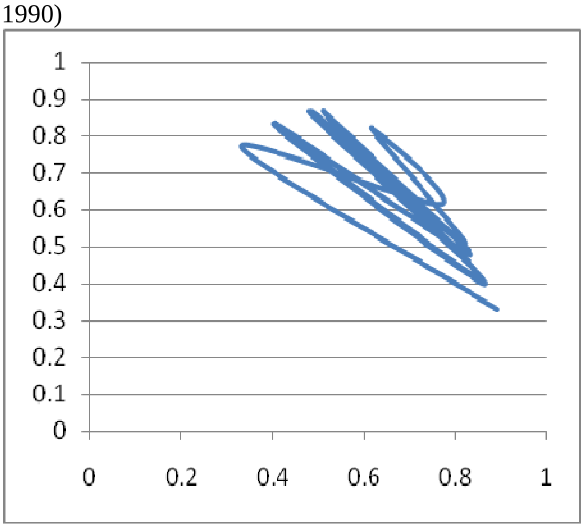

Fig 1.15: LOGISITIC MAP AT r = 3.4691 (1991 - 2020)

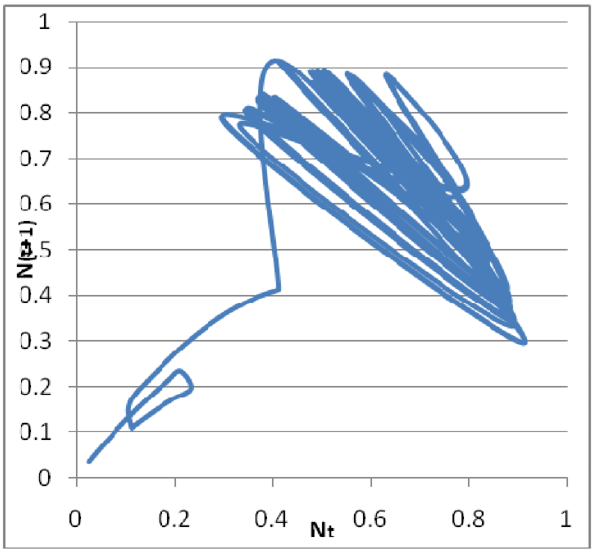

Fig 1.16: A GRAPH OF N(t+1) against N_t (1911 - 2020)

DISCUSSION

The degree of chaos depends on the dimensionless parameter r. From Table 1.3, it can be clearly seen that at carrying capacity (K) of 3 million, the population trend at the early years of the population was in line with the population census data collected till 1952. The difference after 1952 was due to an exponential upshot of population between 1952 and 1963 and till date due to an exponential increase in the growth of the population.

From the time series plot, the population experienced an increase (fig 1.3. and fig 1.4), but the plot showed a decreasing trend. This was because the fixed point (attractor) at this rate (r = 1.0920) was below the initial condition (current population) at that time (0.2329). So the graph tends to decrease towards it attractor. In figure 1.5, the population experienced an increase up to a stable point (attractor) between (r = 1.7) as it converges at 0.411764.

The population began to experience periodic doubling in figure 1.6, figure 1.7 and figure 1.8. This doubling cannot be attributed to any given number of attractors. An N – attractor graph of population doubling implies chaos, which was clearly shown in figure 1.9.

The logistic map vindicated if a population is stable or chaotic. Relating the growth rate of the population to logistic map, it can be seen that in figure 1.10, figure 1.11 and figure 1.12, the plots follow a regular pattern implying that the population was a bit stable. In figure 1.13, due to increasing the growth rate which led to increase in the population, the population becomes chaotic. Figure 1.14 and figure 1.15 showed the same chaotic trend, as the growth rate of the state for the period in which the graph was plotted was still relatively high and would lead to further increase in the population. Figure 1.16 is a summary of figure 1.10 –figure 1.15. From the map(figure 1.16), the graph was linear (implying stable population) at the start before it became periodic (implying chaotic population).

The population can only be brought back to a stable (linear on the map) form if the growth rate is made to be 2.0 or less. For it is at this growth rate ($\leq$ 2.0) that a stable population can be obtained as the logistic map shows a stable (constant) trend which corresponds to a growth rate of 0.04306 i.e 4.31% for real population growth.

With a growth rate of 2.0 on the logistic map, irrespective of the initial value condition, the logistic equation as given below showed a stable population at 0.5, which corresponds to half the carrying capacity (i.e 1.5million) for which chaos cannot occur.

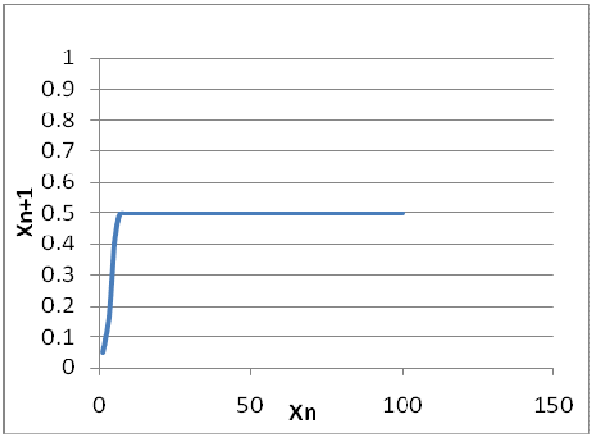

Fig 1.17: A GRAPH OF X_n AGAINST time (years) r = 2.0 and Xo = 0.02459

CONCLUSION

A population is definitely chaotic once the population is more than the carrying capacity, as the individuals of the population suffer inadequate environmental resources needed to sustain a stable population. This implies chaos.

A case of Lagos State in Nigeria with an obtained value of carrying capacity of 3 million and the current population being more than 17 million. The chaotic behaviour was successfully shown using the logistic difference equation on the logistic map.

A graph of N_{t+1} against N_t (1911 – 1920) showed a chaotic behavior and this can only be brought under control by reducing the growth rate to a rate which gives a stable population (i.e. $\leq$ 2.0). This reduction can be obtained by decrease in birth rate, immigration and increase in emigration.

Decrease in birth rate can be brought about by government adopting policies that limits the number of children per family. Also, increase in emigration and decrease in immigration can be achieved by 100% increase in tax for people of the state as against the tax rate of other states, and by increasing employment and standard of living within other states in Nigeria.

REFERENCES

1. Gotelli (1995); A premier of ecology. Sinauer Associates, Sunderland, Mass.
2. Hassell, M. P., Lawton, J. H., May, R. M. (1976). Patterns of dynamical behavior in single-species populations. Journal of Animal Ecology 45, 471–486.
3. Keyfits, N. and Fkeyer, W. (1990): World Population Growth and Ageing; University of Chicago Press
4. Rubinow, A.C. (1975): Simple Mathematical models with complicated Dynamics; Nature and Science.
5. Ibrahim, M.O. and Lewis, I.N. (2007): Population model of Gwer Local Government Area of Benue

State; Proceedings of the Mathematical Association of Nigeria (MAN). Pp 127 – 131.

6. Ogbeide E. M. and Ikpotokin O. (2009) Population Model of Esan West Local Government Area of Edo State, Nigeria. Researcher 1(3), 2009, www.sciencepub.net
7. Lagos State (2006), Population by sex and local government area. Lagos state biannual publication.
8. May, R. M. (1974). Biological populations with non overlapping generations: stable points, stable cycles, and chaos. Science 186, 645–647.
9. May, R. M. (1976). Simple mathematical models with very complicated dynamics.Nature 261, 459–467.