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EXPECTED OVERFLOW FOR STORAGE MODELS WITH POISSON INPUTS, POISSON OUTPUTS AND DETERMINISTIC RELEASE RULE

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ABSTRACT:

In this paper we discuss a finite storage problem with random inputs and random outputs together with linear release policy. For this class of problem the expected amount of overflow is arrived at 3rd order differential equation. Using the imbedding technique the Laplace Transform function for the expected amount of overflow before emptiness is obtained. The same technique is also used to determine the expected amount of overflow with any number of emptiness in this period.

Key words: Finite dam – Integro-differential equation – Separable kernel – Overflow – Emptiness – Absorbing barrier, Reflecting barrier.

Mathematics Subject Classification 60K25; 11Y65.

[I] INTRODUCTION:

Problems with two barriers $X(t) = 0$ and $X(t) = k$ were studied in different contexts. In insurance problems, $X(t)$ is the risk reserve at time *'t'* and the point of interest is the survival period of the company. In neuronal spike trains $X(t)$ corresponds to cumulate potential gathered by the neuron in time *'t'* and one is interested in studying the neuron firing times. In congession problems our interest is to study busy period of the server. In storage problems we are interested in studying the first passage time of emptiness as well as overflow. Keilson [4] has used the compensation function to study the first emptiness period with a linear release policy. In this case the process is skip free in the negative direction. Phatarfod [7] has used Wald [14] Identity to study the emptiness period without overflow or with any number of overflows for Keilson's model. Here $X(t) = k$ is treated as absorbing barrier as well as a reflecting barrier. Here also the input is random and the release rule is taken as linear drift. Puri and Senthuria [8] has considered a dam model of infinite depth with random inputs and random outputs. Tsuri and Osaki [9] in a different context have considered a cumulative process with exponential decay and random outputs and arrived at an integral equation for the first passage time density for overflow. Perry et. al [5] studied the one sided and two sided first exist problem for a class of compound Poisson process with linear deterministic decrease between positive and negative jumps. In another article Perry et. al [6] have studied the expected total discounted cost of switching and maintaining extra capacity as well as the total expected discounted cost of discarded service *M/G/1* queue. Avanzi et. al [1] have studied optimal divided problem to find the dividend payment strategy that maximizes the expected discounted value of dividends which are to be paid to the policy holders until the company is ruined. Vasudevan and Vittal [10] have considered a finite dam with random inputs and random outputs together with a exponential release. Using the imbedding method [2] they have derived closed form solutions for the first passage times of emptiness and overflow when the barriers are absorbing or when one of them is absorbing and the other as reflecting. Vital et. al [11,12,13] considered storage models for expected overflows and FPTD without drift.

In this contribution we will be concerned with a finite storage system with Poisson inputs and Poisson outputs together with linear release policy. The amount of inputs and outputs governed by exponential distributions. The importance of the present work is to show the imbedding method works out well in determining expected amount of overflow in a given time. In section 2, we obtain the expected amount of overflow in time 't' without emptiness and in section 3, we analyze the expected amount of overflow allowing any number of emptiness.

[II]. EXPECTED AMOUNT OF OVERFLOW IN A GIVEN TIME

Define $S(x,k,t)$ as the expected amount of overflow in time *'t'* assuming that there is no emptiness of the dam in time *'t'*.

As derived earlier in all the models the integro-differential equation for $S(x,k,t)$ is

$$\alpha\frac{\partial S}{\partial t}+\alpha\frac{\partial S}{\partial x}(l+\vartheta)S=\vartheta_1\,\eta\int_x^k e^{-\eta(y-x)}S(y,k,t)\;dy+\delta(t)\vartheta_1\eta\int_k^\infty e^{-\eta(y-x)}\;(y-k)dy$$

$$+\vartheta_1\eta\int_k^\infty e^{-\eta(y-x)}\;dyS(k,k,t)+\vartheta_2\,\eta\int_0^x e^{\eta(y-x)}\;S(y,k,t)\;dt \quad (2.1)$$

Here we have to observe that the second integral on the RHS corresponds to the excess overflow in the initial interval of time *'dt'* and the third integral takes care of the process, repeating overflows from the restart level $X=k$

Define

$$\bar{S}(x,k,l)=\int_o^\infty e^{-lt}\,S(x,k,l)\;dt \quad (2.2)$$

Taking Laplace Transform with respect to time *'t'* (2.1) gets converted to

$$\alpha\frac{\partial\bar{S}}{\partial x}+(l+\vartheta)\bar{S} = \vartheta_1\eta\int_x^k e^{-\eta(y-x)}\bar{S}\,(x,k,l)\;dy+\vartheta_1\eta\int_k^\infty e^{-\eta(y-x)}\;(y-k)dy$$
$$+\vartheta_1\eta\int_k^\infty e^{-\eta(y-x)}\;dy\bar{S}\,(k,k,l)+\vartheta_2\eta\int_0^x e^{\eta(y-x)}\;\bar{S}\,(y,k,l)\;dy \qquad (2.3)$$

The differential equation for $\bar{S}\,(x,k,l)$ can be obtained as:

$$\propto\frac{\partial^3\bar{S}}{\partial x^3}+(l+\vartheta)\frac{\partial^2\bar{S}}{\partial x^2}+[\vartheta(1-2p)+\eta^2\;\alpha]\frac{\partial\bar{S}}{\partial x}-\eta^2 l=0 \qquad (2.4)$$

and its general solution is

$$\bar{S}(x,k,l)=A_4e^{m_1x}+B_4e^{m_2x}+C_4e^{m_3x} \qquad (2.5)$$

Where $A_4,\;B_4$ *and* C_4 are functions of ‘*k*’ and ‘*l*’

Inserting the solution (2.5) into the equation (2.3) we get two conditions

$$\eta\left(\frac{A_4m_1e^{m_1k}}{m_1-\eta}+\frac{B_4m_2e^{m_2k}}{m_2-\eta}+\frac{C_4m_3e^{m_3k}}{m_3-\eta}\right)+1=0 \qquad (2.6)$$

$$and \qquad \frac{A_4\,e^{m_1k}}{m_1+\eta}+\frac{B_4e^{m_2k}}{m_2+\eta}+\frac{C_4e^{m_3k}}{m_3+\eta}=0 \qquad (2.7)$$

$\bar{S}(0,k,l)=0$ which implies $\qquad A_4+B_4+C_4=0 \qquad (2.8)$

Equations (2.5), (2.6), (2.7) and (2.8) give the complete closed form solution for $\bar{S}(x,k,l)$, the Laplace transform of the function giving the expected amount of overflow.

[III]. EXPECTED AMOUNT OF OVERFLOW ALLOWING ANY NUMBER OF EMPTINESS

Here we treat $X=0$ as a reflecting barrier we already treated $X=k$ as a reflecting barrier.

Hence in this situation both $X=0$ and $X=k$ are reflecting barriers.

Since the lower barrier can also be reached for emptiness, let us define $S_{11}(x,k,t)$ as the expected amount of overflow before reaching $X=0$ and taking $\bar{S}_{11}(x,k,l)$ as its Laplace Transform, the total equation for $\bar{S}_{11}(x,k,l)$ is:

$$\alpha\frac{\partial\bar{S}_{11}(x,k,l)}{\partial x}+(l+\vartheta)\bar{S}_{11}(x,k,l)$$

$$= \vartheta_1 \eta \int_x^k e^{-\eta(y-x)} \bar{S}_{11}(y,k,l)\ dy + \vartheta_1 \eta \int_k^\infty e^{-\eta(y-x)}\ (y-k)dy$$

$$+\vartheta_1 \eta \int_k^\infty e^{-\eta(y-x)}\ dy\, \bar{S}_{11}(k,k,l) + \vartheta_2 \eta \int_0^x e^{\eta(y-x)}\ \bar{S}_{11}(y,k,l)\ dy \qquad (3.1)$$

The solution for $\bar{S}_{11}(x,k,l)$ is

$$\bar{S}_{11}(x,k,l) = A_{51} e^{m_1 x} + B_{51} e^{m_2 x} + C_{51} e^{m_3 x} \qquad (3.2)$$

subject to the boundary condition

$$\eta\left(\frac{A_{51} m_1 e^{m_1 k}}{m_1 - \eta} + \frac{B_{51} m_2 e^{m_2 k}}{m_2 - \eta} + \frac{C_{51} m_3 e^{m_3 k}}{m_3 - \eta}\right) + 1 = 0 \qquad (3.3)$$

$$and \qquad \frac{A_{51}\ m_1}{m_1 + \eta} + \frac{B_{51} m_2}{m_2 + \eta} + \frac{C_{51} m_3}{m_3 + \eta} = 0 \qquad (3.4)$$

The third boundary condition is given by: $\left.\frac{\partial \bar{S}_{11}(x,k,l)}{\partial x}\right|_{x=0} = 0$

$$A_{51} m_1 + B_{51} m_2 + C_{51} m_3 = 0 \qquad (3.5)$$

Equations (3.2), (3.3), (3.4) and (3.5) can determine the closed form solution for $\bar{S}_{11}(x,k,l)$

Defining $S_{12}(x,k,t)$ as the expected amount of overflow allowing emptiness by crossing and taking $\bar{S}_{12}(x,k,l)$ as its Laplace Transform, the total equation for $\bar{S}_{12}(x,k,l)$ is:

$$\alpha \frac{\partial \bar{S}_{12}(x,k,l)}{\partial x} + (l+\vartheta)\bar{S}_{12}(x,k,l)$$

$$= \vartheta_1 \eta \int_x^k e^{-\eta(y-x)} \bar{S}_{12}(y,k,l)\ dy + \vartheta_1 \eta \int_k^\infty e^{-\eta(y-x)}\ (y-k)dy$$

$$+\vartheta_1 \eta \int_k^\infty e^{-\eta(y-x)}\ dy\, \bar{S}_{12}(k,k,l) + \vartheta_2 \eta \int_0^x e^{\eta(y-x)}\ \bar{S}_{12}(y,k,l)\ dy$$

$$+\vartheta_2 \eta \int_{-\infty}^0 e^{\eta(y-x)}\ dy\, \bar{S}_{12}(0,k,l) \qquad (3.6)$$

The last term allows the process to cross $X = 0$ in initial interval of time '*dt*' and restart the process from $X = 0$ to proceed to overflow between time '*t*' and '*t*+*dt*'

The solution for $\bar{S}_{12}(x,k,l)$ is

$$\bar{S}_{12}(x,k,l) = A_{52}e^{m_1 x} + B_{52}e^{m_2 x} + C_{52}e^{m_3 x} \tag{3.7}$$

subject to the boundary condition

$$\eta\left(\frac{A_{52}m_1e^{m_1 k}}{m_1-\eta} + \frac{B_{52}m_2e^{m_2 k}}{m_2-\eta} + \frac{C_{52}m_3e^{m_3 k}}{m_3-\eta}\right) + 1 = 0 \tag{3.8}$$

and

$$\frac{A_{52}\,m_1}{m_1+\eta} + \frac{B_{52}m_2}{m_2+\eta} + \frac{C_{52}m_3}{m_3+\eta} = 0 \tag{3.9}$$

The third boundary condition is given by $\bar{S}_{12}(0,k,l) = 0$

$$A_{52} + B_{52} + C_{52} = 0 \tag{3.10}$$

Equations (3.7), (3.8), (3.9) and (3.10) can determine the closed form solution for $\bar{S}_{12}(x,k,l)$ Then the amount of expected overflow with any number of emptiness in time *'t'* is the sum $\bar{S}_{11}(x,k,l) + \bar{S}_{12}(x,k,l)$

[IV] CONCLUSION:

In conclusion we point out that the central idea of this paper is to show that the imbedding approach of Richard Bellman works out elegantly to derive a variety of results for a two way Markov jump process with linear drift and constant barriers having the transition density in a separable form. Here, we treated one of the barriers as absorbing and the other as reflecting or both the barriers reflecting. These techniques are carried out to calculate the expected amount of overflow in the storage model. One can only see that the choice of the kernel decides the closed form analytical solution for the FPTD and the expected amount of overflow. If the kernel is chosen in the form

$$k(x,y) = \frac{\lambda^{n+1}}{2n!}|x-y|^n\, e^{-\lambda|x-y|}$$

We will arrive at a differential equation of higher order which is a complicated one to arrive at a closed form solution.

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