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ON PARTIAL SUMS OF MOCK THETA FUNCTIONS OF ORDER 2

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ABSTRACT:
In his last letter to Hardy, Ramanujan defined 17 functions F(q), where $|q| < 1$. He called them mock theta functions, because as q radially approaches any point $e^{2\Pi i\Gamma}$ (r rational), there is a theta function $F_r(q)$ with $F(q) - F_r(q) = 0$ (1). In this paper, we establish relations connecting mock theta functions, partial mock theta functions of order 2 and infinite products analogous to the identities of Ramanujan.

Keywords : Mock theta functions, partial mock theta functions

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[I] INTRODUCTION

Ramanujan's last mathematical creation was his mock theta functions which he discovered during the last years of his life. The first detailed description of these functions was given by Watson in his celebrated Presidential Address delivered at the meeting of the London Mathematical Society in November, 1935.

Ramanujan's general definition of a mock theta function is a function of $f(q)$ defined by a q-series convergent when $|q| < 1$ which satisfies the following two conditions,

(a) For every root ξ of unity, there exist a θ-function θ(q) such that difference between $f(q)$ and θ(q) is bounded as q $\rightarrow \xi$, radially.
(b) There is no single theta function which works for all ξ, i.e. for every θ-function θ(q) there is some root of unity ξ for which $f(q)$ minus the theta function θ(q) is unbounded as q $\rightarrow \xi$ radially.

Ramanujan gave a list of seventeen mock theta functions and labeled them as third, fifth and seventh orders without giving any reason for his classification.

If $$M(q) = \sum_{n=0}^{\infty} \Omega_n \tag{1.1}$$

is a mock theta function, then the corresponding partial mock theta function is denoted by the terminating series,

$$M_r(q) = \sum_{n=0}^{r} \Omega_n \tag{1.2}$$

A study of these sums and expansions has been made by Watson [21], Agarwal [1] and Andrews [2]. Later on, Andrews and Hickerson [3], Choi [6] and Gordon and Mc Intosh [11] studied certain q-series in the Lost Notebook and named them as sixth, eighth and tenth order mock theta functions.Also, relations connecting mock theta functions and partial mock theta functions are given by Srivastava [20] and Denis *et al.* [7].

<u>Mock theta functions of order 2 are;</u>

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}.(-q;q^2)_n}{(-q^4;q^4)_n} \tag{1.3}$$

$$B(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}.(-q;q^2)_n}{(-q^2;q^4)_{n+1}} \tag{1.4}$$

$$C(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}.(-q;q^2)_n}{(q;q^2)_{n+1}} \tag{1.5}$$

[Mc Intosh [15]]

<u>Partial mock theta functions of order 2 are;</u>

$$A_m(q) = \sum_{n=0}^{m} \frac{q^{n^2}.(-q;q^2)_n}{(-q^4;q^4)_n} \tag{1.6}$$

$$B_m(q) = \sum_{n=0}^{m} \frac{q^{(n+1)^2}.(-q;q^2)_n}{(-q^2;q^4)_{n+1}} \tag{1.7}$$

$$C_m(q) = \sum_{n=0}^{m} \frac{q^{(n+1)^2}.(-q;q^2)_n}{(q;q^2)_{n+1}} \tag{1.8}$$

Ramanujan, in chapter 16 of his second notebook defined theta functions as follows;

$$\chi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.9}$$

[Ramanujan [17] and Berndt [5]]

An identity due to Euler is,

$$\sum_{n=0}^{\infty} \frac{x^n q^{\binom{n}{2}}}{(q;q)_n} = (-x;q)_\infty \tag{1.10}$$

[Euler [8]; chap. 16]

The special cases of the above identity are ;

$$L(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} = \frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty} \tag{1.11}$$

$$T(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} = \frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty} \tag{1.12}$$

The Jackson – Slater identity;

Jackson [14] discovered the following identity;

$$U(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty} \tag{1.13}$$

This identity was independently rediscovered by Slater [Slater [19]; Eqn.(39)] who also discovered its companion identity [Slater [19]; Eqn.(38)]

$$V(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty} \tag{1.14}$$

The Famous Roger's –Ramanujan identities are,

$$M(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty} \tag{1.15}$$

$$N(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty} \qquad (1.16)$$

[Rogers [18] and Ramanujan [16]]

Two identities analogous to the Rogers- Ramanujan identities are the so-called Gollnitz – Gordon identities given by,

$$E(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty} \qquad (1.17)$$

$$\eta(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty} \qquad (1.18)$$

[Gordon [10] and Gollnitz [9]]

Hahn [12] & [13] defined the septic analogues of the Rogers- Ramanujan functions as

$$X(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \qquad (1.19)$$

$$Y(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \qquad (1.20)$$

$$Z(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \qquad (1.21)$$

The nonic analogues of Rogers – Ramanujan functions are

$$O(q) = \sum_{n=0}^{\infty} \frac{(q;q)_{3n}.q^{3n^2}}{(q^3;q^3)_n(q^3;q^3)_{2n}} = \frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \qquad (1.22)$$

$$Q(q) = \sum_{n=0}^{\infty} \frac{(q;q)_{3n}(1-q^{3n+2})q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} = \frac{(q^2,q^7,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \qquad (1.23)$$

$$W(q) = \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1}.q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} = \frac{(q,q^8,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \qquad (1.24)$$

These equalities are due to Bailey [Bailey [4]; Eqn.(1.6),(1.7) and (1.8)]

The partial sums of the above functions are given as follows,

$$\chi_m(q) = \sum_{n=0}^{m} q^{n(n+1)/2} \tag{1.25}$$

$$L_m(q) = \sum_{n=0}^{m} \frac{q^{n^2}}{(q^2;q^2)_n} \tag{1.26}$$

$$T_m(q) = \sum_{n=0}^{m} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \tag{1.27}$$

$$U_m(q) = \sum_{n=0}^{m} \frac{q^{2n^2}}{(q;q)_{2n}} \tag{1.28}$$

$$V_m(q) = \sum_{n=0}^{m} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} \tag{1.29}$$

$$M_m(q) = \sum_{n=0}^{m} \frac{q^{n^2}}{(q;q)_n} \tag{1.30}$$

$$N_m(q) = \sum_{n=0}^{m} \frac{q^{n(n+1)}}{(q;q)_n} \tag{1.31}$$

$$E_m(q) = \sum_{n=0}^{m} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} \tag{1.32}$$

$$\eta_m(q) = \sum_{n=0}^{m} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} \tag{1.33}$$

$$X_m(q) = \sum_{n=0}^{m} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} \tag{1.34}$$

$$Y_m(q) = \sum_{n=0}^{m} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} \tag{1.35}$$

$$Z_m(q) = \sum_{n=0}^{m} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} \tag{1.36}$$

$$O_m(q) = \sum_{n=0}^{m} \frac{(q;q)_{3n}.q^{3n^2}}{(q^3;q^3)_n(q^3;q^3)_{2n}} \tag{1.37}$$

$$Q_m(q) = \sum_{n=0}^{m} \frac{(q;q)_{3n}(1-q^{3n+2})q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} \tag{1.38}$$

$$W_m(q) = \sum_{n=0}^{m} \frac{(q;q)_{3n+1}.q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} \tag{1.39}$$

[II] METHODOLOGY

Srivastava [20] gave the following identity;

$$\sum_{m=0}^{\infty} \delta_m \sum_{r=0}^{m} \alpha_r = \left(\sum_{r=0}^{\infty}\alpha_r\right)\left(\sum_{m=0}^{\infty} \delta_m\right) - \sum_{r=0}^{\infty} \alpha_{r+1} \sum_{m=0}^{r} \delta_m \tag{2.1}$$

He assumed $\delta_m = \dfrac{(aq-e)(e-bq)(a,b)_m q^m}{(q-e)(e-abq)(e,\frac{abq^2}{e})_m}$ and gave different values for α_r to get a number of partial mock theta function identities. The same method is followed in present investigation and the values of δ_m are chosen as infinite product representation of mock theta functions and the values of α_r are assumed as mock theta functions of different orders.

After substituting the values of δ_m and α_r in the above identity, we obtain a number of partial mock theta function identities.

[III] RESULTS

We shall make use of the known identity of Srivastava [20] to obtain relations connecting mock theta functions and partial mock theta functions of order 2.

A) Taking $\delta_m = q^{m(m+1)/2}$ in (2.1) and by (1.9) & (1.25), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty} q^{m(m+1)/2}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}.\chi_m(q). \quad (3.1)$$

i) Taking $\alpha_r = \frac{q^{r^2}(-q;q^2)_r}{(-q^4;q^4)_r}$ in (3.1) and making use of (1.3) and (1.6), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.\sum_{r=0}^{\infty}\frac{q^{r^2}(-q;q^2)_r}{(-q^4;q^4)_r}$$

$$= \sum_{m=0}^{\infty} q^{m(m+1)/2}.\sum_{r=0}^{m}\frac{q^{r^2}(-q;q^2)_r}{(-q^4;q^4)_r} + \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(-q^4;q^4)_{r+1}}.\chi_m(q).$$

$$\Rightarrow \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.\mathrm{A(q)} = \sum_{m=0}^{\infty} q^{m(m+1)/2}.\mathrm{A}_m\mathrm{(q)} + \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(-q^4;q^4)_{r+1}}.\chi_m(q). \quad (3.2)$$

ii) Taking $\alpha_r = \frac{q^{(r+1)^2}(-q;q^2)_r}{(-q^2;q^4)_{r+1}}$ in (3.1) and making use of (1.4) and (1.7), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.\mathrm{B(q)} = \sum_{m=0}^{\infty} q^{m(m+1)/2}.\ \mathrm{B_m(q)} + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(-q^2;q^4)_{r+2}}\chi_m(q). \quad (3.3)$$

iii) Taking $\alpha_r = \frac{q^{(r+1)^2}(-q;q^2)_r}{(q;q^2)_{r+1}}$ in (3.1) and making use of (1.5) and (1.8),

we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.\mathrm{C(q)} = \sum_{m=0}^{\infty} q^{m(m+1)/2}.\mathrm{C_m(q)} + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}}\chi_m(q) \quad (3.4)$$

B) Taking $\delta_m = \frac{q^{m^2}}{(q^2;q^2)_m}$ in (2.1) and by (1.11) & (1.26), we get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}\ \mathrm{L_m(q)} \quad (3.5)$$

i) Taking $\alpha_r = \frac{q^{r^2}(-q;q^2)_r}{(-q^4;q^4)_r}$ in (3.5) and making use of (1.3) and (1.6), we get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty}.A(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q^2;q^2)_m}.A_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(-q^4;q^4)_{r+1}}.L_m(q) \quad (3.6)$$

ii) Taking $\alpha_r = \frac{q^{(r+1)^2}(-q;q^2)_r}{(-q^2;q^4)_{r+1}}$ in (3.5) and making use of (1.4) and (1.7), we get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty}.B(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q^2;q^2)_m}.B_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(-q^2;q^4)_{r+2}}.L_m(q) \quad (3.7)$$

iii) Taking $\alpha_r = \frac{q^{(r+1)^2}(-q;q^2)_r}{(q;q^2)_{r+1}}$ in (3.5) and making use of (1.5) and (1.8), we get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty}C(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q^2;q^2)_m}.C_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+2)^2}.(-q;q^2)_{r+1}}{(q;q^2)_{r+2}}.L_m(q) \quad (3.8)$$

C) Taking $\delta_m = \frac{q^{m(m+1)}}{(q^2;q^2)_m}$ in (2.1) and by (1.12) & (1.27), we get

$$\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(q^2;q^2)_m}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}T_m(q). \quad (3.9)$$

i) Taking $\alpha_r = \frac{q^{r^2}(-q;q^2)_r}{(-q^4;q^4)_r}$ in (3.9) and making use of (1.3) and (1.6), we get

$$\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty}A(q) = \sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(q^2;q^2)_m}.A_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(-q^4;q^4)_{r+1}}.T_m(q) \quad (3.10)$$

ii) Taking $\alpha_r = \frac{q^{(r+1)^2}(-q;q^2)_r}{(-q^2;q^4)_{r+1}}$ in (3.9) and making use of (1.4) and (1.7), we get

$$\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty}B(q) = \sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(q^2;q^2)_m}B_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(-q^2;q^4)_{r+2}}.T_m(q) \quad (3.11)$$

iii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}(-q;q^2)_r}{(q;q^2)_{r+1}}$ in (3.9) and making use of (1.5) and (1.8), we get

$$\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty}\text{C(q)} = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q^2;q^2)_m}.\text{C}_\text{m}(\text{q}) + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}}.T_m(q) \qquad (3.12)$$

D) Taking $\delta_m = \dfrac{q^{m^2}}{(q;q)_m}$ in (2.1) and by (1.15) & (1.30), we get

$$\frac{1}{(q,q^4;q^5)_\infty}.\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}.\text{M}_\text{m}(\text{q}) \qquad (3.13)$$

i) Taking $\alpha_r = \dfrac{q^{r^2}(-q;q^2)_r}{(-q^4;q^4)_r}$ in (3.13) and by (1.3) and (1.6), we get

$$\frac{1}{(q,q^4;q^5)_\infty}.\text{A(q)} = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}.\text{A}_\text{m}(\text{q}) + \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(-q^4;q^4)_{r+1}}.M_m(q) \qquad (3.14)$$

ii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}(-q;q^2)_r}{(-q^2;q^4)_{r+1}}$ in (3.13) and by (1.4) and (1.7), we get

$$\frac{1}{(q,q^4;q^5)_\infty}.\text{B(q)} = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}.\text{B}_\text{m}(\text{q}) + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(-q^2;q^4)_{r+2}}.M_m(q) \qquad (3.15)$$

iii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}(-q;q^2)_r}{(q;q^2)_{r+1}}$ in (3.13) and by (1.5) and (1.8), we get

$$\frac{1}{(q,q^4;q^5)_\infty}.\text{C(q)} = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}.\text{C}_\text{m}(\text{q}) + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}}.M_m(q) \qquad (3.16)$$

E) Taking $\delta_m = \dfrac{q^{m(m+1)}}{(q;q)_m}$ in (2.1) and by (1.16) & (1.31), we get

$$\frac{1}{(q^2,q^3;q^5)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q;q)_m}.\sum_{r=0}^{m}\alpha_r.+\sum_{r=0}^{\infty}\alpha_{r+1}.N_m(q) \qquad (3.17)$$

i) Taking $\alpha_r = \frac{q^{r(r+1)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (3.17) and by (1.3) and (1.6), we get

$$\frac{1}{(q^2,q^3;q^5)_\infty}.P(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q;q)_m}.P_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}.N_m(q) \qquad (3.18)$$

ii) Taking $\alpha_r = \frac{(-1)^r.q^{(r+1)^2}(q;q^2)_r}{(-q;q)_{2r+1}}$ in (3.17) and by (1.4) and (1.7), we get

$$\frac{1}{(q^2,q^3;q^5)_\infty}R(q) = \sum_{m=0}^{\infty}R_m(q).\frac{q^{m(m+1)}}{(q;q)_m} + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}.q^{(r+2)^2}(q;q^2)_{r+1}}{(-q;q)_{2r+3}}.N_m(q) \qquad (3.19)$$

iii) Taking $\alpha_r = \frac{(-1)^r.q^r(q;q^2)_r}{(-q;q)_r}$ in (3.17) and by (1.5) and (1.8), we get

$$\frac{1}{(q^2,q^3;q^5)_\infty}S(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q;q)_m}.S_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}.q^{r+1}(q;q^2)_{r+1}}{(-q;q)_{r+1}}.N_m(q) \qquad (3.20)$$

F) Taking $\delta_m = \frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}$ in (2.1) and by (1.19) & (1.34), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}.X_m(q) \qquad (3.21)$$

i) Taking $\alpha_r = \frac{q^{r^2}(-q;q^2)_r}{(-q^4;q^4)_r}$ in (3.21) and by (1.3) and (1.6), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.A(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}.A_m(q)$$

$$+\sum_{r=0}^{\infty}\frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(-q^4;q^4)_{r+1}}.X_m(q) \qquad (3.22)$$

ii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}(-q;q^2)_r}{(-q^2;q^4)_{r+1}}$ in (3.21) and by (1.4) and (1.7), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.\mathrm{B(q)} = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}.+\sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(-q^2;q^4)_{r+2}}.X_m(q) \quad (3.23)$$

iii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}(-q;q^2)_r}{(q;q^2)_{r+1}}$ in (3.21) and by (1.5) and (1.8), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\mathrm{C(q)} = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}.+\sum_{r=0}^{\infty}\frac{q^{(r+2)^2}.(-q;q^2)_{r+1}}{(q;q^2)_{r+2}}.X_m(q) \quad (3.24)$$

G) Take $\delta_m = \dfrac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}$ in (2.1) and by (1.20) & (1.35), we get

$$\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}.\mathrm{Y_m(q)} \quad (3.25)$$

i) Taking $\alpha_r = \dfrac{q^{r^2}}{(-q;q)_r}$ in (3.25) and by (1.3) and (1.6), we get

$$\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.\mathrm{F(q)} = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}.\mathrm{F_m(q)} + \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}}{(-q;q)_{r+1}}.Y_m(q) \quad (3.26)$$

ii) Taking $\alpha_r = \dfrac{q^{r(r+1)}}{(-q;q)_r}$ in (3.25) and by (1.4) and (1.7), we get

$$\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.\mathrm{G(q)} = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}.\mathrm{G_m(q)} + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)}}{(-q;q)_{r+1}}.Y_m(q) \quad (3.27)$$

iii) Taking $\alpha_r = \dfrac{q^{2r^2}}{(q;q^2)_r}$ in (3.25) and by (1.5) and (1.8), we get

$$\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.\mathrm{H(q)} = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}.\mathrm{H_m(q)} + \sum_{r=0}^{\infty}\frac{q^{2(r+1)^2}}{(q;q^2)_{r+1}}.Y_m(q) \quad (3.28)$$

H) Taking $\delta_m = \dfrac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}$ in (2.1) and by (1.21) & (1.36), we get

$$\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}.Z_m(q). \quad (3.29)$$

i) Taking $\alpha_r = \dfrac{q^{r^2}}{(q^{r+1};q)_r}$ in (3.29) and by (1.3) and (1.6), we get

$$\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.\alpha(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}.\alpha_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}}{(q^{r+2};q)_{r+1}}.Z_m(q) \quad (3.30)$$

ii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}}{(q^{r+1};q)_{r+1}}$ in (3.29) and by (1.4) and (1.7), we get

$$\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.\beta(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}.\beta_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}}{(q^{r+2};q)_{r+2}}.Z_m(q) \quad (3.31)$$

iii) Taking $\alpha_r = \dfrac{q^{r(r+1)}}{(q^{r+1};q)_{r+1}}$ in (3.29) and by (1.5) and (1.8), we get

$$\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.\gamma(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}.\gamma_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)}}{(q^{r+2};q)_{r+2}}.Z_m(q) \quad (3.32)$$

I) Taking $\delta_m = \dfrac{(-q;q^2)_m q^{m^2}}{(q^2;q^2)_m}$ in (2.1) and by (1.17) & (1.32), we get

$$\frac{1}{(q,q^4,q^7;q^8)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{(-q;q^2)_m.q^{m^2}}{(q^2;q^2)_m}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}.E_m(q). \quad (3.33)$$

i) Taking $\alpha_r = \dfrac{q^{r^2}(-q;q^2)_r}{(-q^4;q^4)_r}$ in (3.33) and by (1.3) and (1.6), we get

$$\frac{1}{(q,q^4,q^7;q^8)_\infty}.A(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}(-q;q^2)_m}{(q^2;q^2)_m}.A_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(-q^4;q^4)_{r+1}}.E_m(q) \quad (3.34)$$

ii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}(-q;q^2)_r}{(-q^2;q^4)_{r+1}}$ in (3.33) and by (1.4) and (1.7), we get

$$\frac{1}{(q,q^4,q^7;q^8)_\infty}\text{B(q)} = \sum_{m=0}^{\infty}\frac{q^{m^2}(-q;q^2)_m}{(q^2;q^2)_m}.\text{B}_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(-q^2;q^4)_{r+2}}.E_m(q) \qquad (3.35)$$

iii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}(-q;q^2)_r}{(q;q^2)_{r+1}}$ in (3.33) and by (1.5) and (1.8), we get

$$\frac{1}{(q,q^4,q^7;q^8)_\infty}.\text{C(q)} = \sum_{m=0}^{\infty}\frac{q^{m^2}(-q;q^2)_m}{(q^2;q^2)_m}.\text{C}_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}}.E_m(q) \qquad (3.36)$$

J) Taking $\delta_m = \dfrac{q^{m^2+2m}(-q;q^2)_m}{(q^2;q^2)_m}$ in (2.1) and by (1.18) & (1.33), we get

$$\frac{1}{(q^3,q^4,q^5;q^8)_\infty}.\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{m^2+2m}(-q;q^2)_m}{(q^2;q^2)_m}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}\,\eta_m(q). \qquad (3.37)$$

i) Taking $\alpha_r = \dfrac{q^{r^2}}{(-q;q)_r}$ in (3.37) and by (1.3) and (1.6), we get

$$\frac{1}{(q^3,q^4,q^5;q^8)_\infty}\text{F(q)} = \sum_{m=0}^{\infty}\frac{q^{m^2+2m}(-q;q^2)_m}{(q^2;q^2)_m}.\text{F}_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}}{(-q;q)_{r+1}}.\eta_m(q). \qquad (3.38)$$

ii) Taking $\alpha_r = \dfrac{q^{r(r+1)}}{(-q;q)_r}$ in (3.37) and by (1.4) and (1.7), we get

$$\frac{1}{(q^3,q^4,q^5;q^8)_\infty}\text{G(q)} = \sum_{m=0}^{\infty}\frac{q^{m^2+2m}(-q;q^2)_m}{(q^2;q^2)_m}.\text{G}_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)}}{(-q;q)_{r+1}}.\eta_m(q). \qquad (3.39)$$

iii) Taking $\alpha_r = \dfrac{q^{2r^2}}{(q;q^2)_r}$ in (3.37) and by (1.5) and (1.8), we get

$$\frac{1}{(q^3,q^4,q^5;q^8)_\infty}\text{H(q)} = \sum_{m=0}^{\infty}\frac{q^{m^2+2m}(-q;q^2)_m}{(q^2;q^2)_m}.\text{H}_m(q) + \sum_{r=0}^{\infty}\frac{q^{2(r+1)^2}}{(q;q^2)_{r+1}}.\eta_m(q). \qquad (3.40)$$

K) Taking $\delta_m = \dfrac{q^{2m^2}}{(q;q)_{2m}}$ in (2.1) and by (1.13) & (1.28), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}.\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}.\mathrm{U_m(q)}. \tag{3.41}$$

i) Taking $\alpha_r = \dfrac{q^{r^2}}{(q^{r+1};q)_r}$ in (3.41) and by (1.3) and (1.6), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}.\alpha(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}\alpha_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}}{(q^{r+2};q)_{r+1}}.U_m(q) \tag{3.42}$$

ii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}}{(q^{r+1};q)_{r+1}}$ in (3.41) and by (1.4) and (1.7), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}.\beta(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}\beta_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}}{(q^{r+2};q)_{r+2}}.U_m(q) \tag{3.43}$$

iii) Taking $\alpha_r = \dfrac{q^{r(r+1)}}{(q^{r+1};q)_{r+1}}$ in (3.41) and by (1.5) and (1.8), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}.\gamma(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}\gamma_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)}}{(q^{r+2};q)_{r+2}}.U_m(q) \tag{3.44}$$

L) Taking $\delta_m = \dfrac{q^{2m(m+1)}}{(q;q)_{2m+1}}$ in (2.1) and by (1.14) & (1.29), we get

$$\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty}.\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q;q)_{2m+1}}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}\ \mathrm{V_m(q)} \tag{3.45}$$

i) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}}{(q;q^2)_{r+1}}$ in (3.45) and by (1.3) and (1.6), we get

$$\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\mathrm{I(q)} = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q;q)_{2m+1}}.\mathrm{I_m(q)} + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)/2}}{(q;q^2)_{r+2}}.V_m(q) \tag{3.46}$$

ii) Taking $\alpha_r = \dfrac{q^{(r+1)(r+2/2}}{(q;q^2)_{r+1}}$ in (3.45) and by (1.4) and (1.7), we get

$$\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\text{J(q)} = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q;q)_{2m+1}}.\text{J}_m\text{(q)} + \sum_{r=0}^{\infty}\frac{q^{(r+2)(r+3)/2}}{(q;q^2)_{r+2}}.V_m(q) \quad (3.47)$$

iii) Taking $\alpha_r = \frac{(-1)^r.q^{r^2}}{(-q;q)_{2r}}$ in (3.45) and by (1.5) and (1.8), we get

$$\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\text{K(q)} = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q;q)_{2m+1}}.\text{K}_m\text{(q)} + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}.q^{(r+1)^2}}{(-q;q)_{2(r+1)}}.V_m(q) \quad (3.48)$$

M) Taking $\delta_m = \frac{q^{3m^2}(q;q)_{3m}}{(q^3;q^3)_m(q^3;q^3)_{2m}}$ in (2.1) and by (1.22) & (1.37), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}.\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{3m^2}(q;q)_{3m}}{(q^3;q^3)_m(q^3;q^3)_{2m}}.\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}\text{O}_m\text{(q)} \quad (3.49)$$

i) Taking $\alpha_r = \frac{q^{r^2}(-q;q^2)_r}{(-q^4;q^4)_r}$ in (3.49) and by (1.3) and (1.6), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\text{A(q)} = \sum_{m=0}^{\infty}\frac{q^{3m^2}(q;q)_{3m}}{(q^3;q^3)_m(q^3;q^3)_{2m}}.\text{A}_m\text{(q)}+ \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(-q^4;q^4)_{r+1}}.O_m(q)$$

(3.50)

ii) Taking $\alpha_r = \frac{q^{(r+1)^2}(-q;q^2)_r}{(-q^2;q^4)_{r+1}}$ in (3.49) and by (1.4) and (1.7), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\text{B(q)} = \sum_{m=0}^{\infty}\frac{q^{3m^2}(q;q)_{3m}}{(q^3;q^3)_m(q^3;q^3)_{2m}}.\text{B}_m\text{(q)}+ \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(-q^2;q^4)_{r+2}}.O_m(q)$$

(3.51)

iii) Taking $\alpha_r = \frac{q^{(r+1)^2}(-q;q^2)_r}{(q;q^2)_{r+1}}$ in (3.49) and by (1.5) and (1.8), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\text{C(q)} = \sum_{m=0}^{\infty}\frac{q^{3m^2}(q;q)_{3m}}{(q^3;q^3)_m(q^3;q^3)_{2m}}.\text{C}_m\text{(q)} +$$

$$\sum_{r=0}^{\infty}\frac{q^{(r+2)^2}.(-q;q^2)_{r+1}}{(q;q^2)_{r+2}}.O_m(q) \quad (3.52)$$

N) Taking $\delta_m = \dfrac{(q;q)_{3m}(1-q^{3m+2}).q^{3m(m+1)}}{(q^3;q^3)_m(q^3;q^3)_{2m+1}}$ in (2.1) and by (1.23) & (1.38), we get

$$\frac{(q^2,q^7,q^9;q^9)_\infty}{(q^3;q^3)_\infty}.\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}(1-q^{3m+2}).q^{3m(m+1)}}{(q^3;q^3)_m(q^3;q^3)_{2m+1}} \text{ x } \sum_{r=0}^{m}\alpha_r$$

$$+\sum_{r=0}^{\infty}\alpha_{r+1}\,\mathrm{Q_m(q)} \qquad (3.53)$$

i) Taking $\alpha_r = \dfrac{q^{r^2}}{(-q;q)_r}$ in (3.53) and by (1.3) and (1.6), we get

$$\frac{(q^2,q^7,q^9;q^9)_\infty}{(q^3;q^3)_\infty}.\mathrm{F(q)} = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}(1-q^{3m+2}).q^{3m(m+1)}}{(q^3;q^3)_m(q^3;q^3)_{2m+1}}\,\mathrm{F_m(q)}$$

$$+\sum_{r=0}^{\infty}\frac{q^{(r+1)^2}}{(-q;q)_{r+1}}.Q_m(q) \qquad (3.54)$$

ii) Taking $\alpha_r = \dfrac{q^{r(r+1)}}{(-q;q)_r}$ in (3.53) and by (1.4) and (1.7), we get

$$\frac{(q^2,q^7,q^9;q^9)_\infty}{(q^3;q^3)_\infty}.\mathrm{G(q)} = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}(1-q^{3m+2}).q^{3m(m+1)}}{(q^3;q^3)_m(q^3;q^3)_{2m+1}}\,\mathrm{G_m(q)}$$

$$+\sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)}}{(-q;q)_{r+1}}.Q_m(q) \qquad (3.55)$$

iii) Taking $\alpha_r = \dfrac{q^{2r^2}}{(q;q^2)_r}$ in (3.53) and by (1.5) and (1.8), we get

$$\frac{(q^2,q^7,q^9;q^9)_\infty}{(q^3;q^3)_\infty}.\mathrm{H(q)} = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}(1-q^{3m+2}).q^{3m(m+1)}}{(q^3;q^3)_m(q^3;q^3)_{2m+1}}\,\mathrm{H_m(q)}$$

$$+\sum_{r=0}^{\infty}\frac{q^{2(r+1)^2}}{(q;q^2)_{r+1}}.Q_m(q) \qquad (3.56)$$

O) Taking $\delta_m = \dfrac{q^{3m(m+1)}(q;q)_{3m+1}}{(q^3;q^3)_m (q^3;q^3)_{2m+1}}$ in (2.1) and by (1.24) & (1.39), we get

$$\frac{(q,q^8,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{3m(m+1)}(q;q)_{3m+1}}{(q^3;q^3)_m (q^3;q^3)_{2m+1}}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}\,.\,W_m(q) \quad (3.57)$$

i) Taking $\alpha_r = \dfrac{q^{r^2}}{(q^{r+1};q)_r}$ in (3.57) and by (1.3) and (1.6), we get

$$\frac{(q,q^8,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\alpha(q) = \sum_{m=0}^{\infty}\frac{q^{3m(m+1)}(q;q)_{3m+1}}{(q^3;q^3)_m (q^3;q^3)_{2m+1}}\alpha_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)^2}}{(q^{r+2};q)_{r+1}}.W_m(q) \quad (3.58)$$

ii) Taking $\alpha_r = \dfrac{q^{(r+1)^2}}{(q^{r+1};q)_{r+1}}$ in (3.57) and by (1.4) and (1.7), we get

$$\frac{(q,q^8,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\beta(q) = \sum_{m=0}^{\infty}\frac{q^{3m(m+1)}(q;q)_{3m+1}}{(q^3;q^3)_m (q^3;q^3)_{2m+1}}\beta_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+2)^2}}{(q^{r+2};q)_{r+2}}.W_m(q) \quad (3.59)$$

iii) Taking $\alpha_r = \dfrac{q^{r(r+1)}}{(q^{r+1};q)_{r+1}}$ in (3.57) and by (1.5) and (1.8), we get

$$\frac{(q,q^8,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\gamma(q) = \sum_{m=0}^{\infty}\frac{q^{3m(m+1)}(q;q)_{3m+1}}{(q^3;q^3)_m (q^3;q^3)_{2m+1}}\gamma_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)}}{(q^{r+2};q)_{r+2}}.W_m(q) \quad (3.60)$$

[IV] CONCLUSION

In this paper, we have established relations connecting mock theta functions of order 2 and its partial sums. In the similar way, many results can be established between mock theta functions of different orders and their corresponding partial sums.

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