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INFINITE PRODUCT EXPANSION OF A CONTINUED FRACTION

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ABSTRACT:

The power series $P = \sum_{n=0}^{\infty} a_n q^n$ is defined as $P = P_0 + P_1 + \ldots + P_{m-1}$, where $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$ is a infinite product. Some continued fractions can be represented in terms of power series and infinite products. These infinite products of the continued fractions can be expanded. In this paper, an attempt is made to represent a continued fraction in terms of infinite product expansion.

Keywords: Continued fraction, power series, infinite product expansion

[I] INTRODUCTION

Rogers-Ramanujan's continued fraction is given by

$$R(q) = 1 + \frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\ldots\ldots\ldots\ldots = \frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}$$

Andrews [2] and Hirschhorn [4] have given the 2 dissection and 5 dissection of R(q) and $R(q)^{-1}$ of the above continued fraction. Also Hirschhorn [4] has given the 4 dissection for the above continued fraction.

The following is the Ramanujan-Gollnitz-Gordon's continued fraction (Andrews, [1]);

$$S(q) = 1 + \frac{q+q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^3+q^6}{1+}\,\frac{q^8}{1+}\,\ldots\ldots\ldots\ldots \tag{1.1}$$

Denis *et al.* [3] proved that,

$$S(q) = 1 + \frac{q+q^2}{1+}\ \frac{q^4}{1+}\ \frac{q^3+q^6}{1+}\ \frac{q^8}{1+} \cdots\cdots\cdots = \frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty} \tag{1.2}$$

$$S(q)^{-1} = \frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty} \tag{1.3}$$

In this paper, we give the infinite product expansion for the above continued fractions.

[II] METHODOLOGY

Suppose that |q| < 1, where q is a non-zero complex number, this condition ensures that all the infinite products that we use will converge. We will use the notation,

$$[z;q]_\infty = (z;q)_\infty\,(z^{-1}q;q)_\infty \text{ (for } z\neq 0) \tag{2.1}$$

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n) \tag{2.2}$$

and often we write

$$[z_1,z_2,\ldots\ldots\ldots\ldots,z_n;q]_\infty = [z_1;q]_\infty[z_2;q]_\infty\ldots\ldots[z_n;q]_\infty$$

The following facts can be easily verified;

$$[z^{-1};q]_\infty = -z^{-1}[z;q]_\infty = [zq;q]_\infty \tag{2.3}$$

$$[z,zq;q^2]_\infty = [z;q]_\infty \tag{2.4}$$

$$[z,-z;q]_\infty = [z^2;q^2]_\infty \tag{2.5}$$

$$[z^{-1}q;q]_\infty = [z;q]_\infty \tag{2.6}$$

$$[-1;q]_\infty[q;q^2]_\infty = 2 \tag{2.7}$$

(2.7) is called the Euler's theorem. The theorem states, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

By making use of the above results, we get

$$S(q) = \frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty} = \frac{[q^3;q^8]_\infty}{[q;q^8]_\infty} = \frac{[q^3,q^5;q^{16}]_\infty}{[q,q^7;q^{16}]_\infty} \tag{2.8}$$

$$S(q)^{-1} = \frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty}$$

$$= \frac{[q;q^8]_\infty}{[q^3;q^8]_\infty} = \frac{[q,q^7;q^{16}]_\infty}{[q^3,q^5;q^{16}]_\infty} \tag{2.9}$$

Also, we have the following general relation;

Suppose $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n \in C \setminus \{0\}$ satisfy

i) $a_i \neq q^n a_j$ for $i \neq j$ and any $n \in Z$

ii) $a_1 a_2 \ldots a_n = b_1 b_2 \ldots b_n$

then,

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n} [a_i^{-1} b_j ; q]_\infty}{\prod_{j=1, j\neq i}^{n} [a_i^{-1} a_j ; q]_\infty} = 0 \tag{2.10}$$

This theorem appears without proof as given by Richard Lewis and Zhi-Guo Liu [6], Slater [7] and with a proof as given by Lewis [5].

The m-dissection of the power series $P = \sum_{n=0}^{\infty} a_n q^n$ is the presentation of P as

$P = P_0 + P_1 + \ldots + P_{m-1}$, where $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$

[III] RESULTS

Let us establish the infinite product expansion for the continued fractions S(q) & $S(q)^{-1}$

Theorem 1;

$$\begin{aligned} S(q) = {} & \frac{q^2[q^2,q^4,q^5;q^{16}]_\infty}{[q^6,q^9;q^{16}]_\infty [q^4,q^{16};q^{32}]_\infty} - \frac{q^3[q,q^2,q^4;q^{16}]_\infty [q^{10};q^{32}]_\infty}{[q^3,q^6;q^{16}]_\infty [q^4,q^{14},q^{16};q^{32}]_\infty} \\ & + \frac{[q^4,q^5,q^7;q^{16}]_\infty}{[q^3,q^9;q^{16}]_\infty [q^4,q^{20};q^{32}]_\infty} - \frac{q[q^3,q^8;q^{16}]_\infty [q^{10};q^{32}]_\infty}{[q^4,q^5;q^{16}]_\infty [-1;q^{16}]_\infty [q^{14},q^{16};q^{32}]_\infty} \\ & + \frac{q[q^4,q^5;q^{16}]_\infty [q^6;q^{32}]_\infty}{[q^3,q^8;q^{16}]_\infty [-1;q^{16}]_\infty [q^8,q^{14};q^{32}]_\infty} \\ & + \frac{q[q^3,q^4;q^{16}]_\infty [q^{10},q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty [q^4,q^8,q^{14},q^{16};q^{32}]_\infty} \end{aligned} \tag{3.1}$$

and

$$S(q)^{-1} = \frac{q^2[q^3,q^4;q^{16}]_\infty [q^2;q^{32}]_\infty}{[q^7;q^{16}]_\infty [q^4,q^6,q^{16};q^{32}]_\infty} - \frac{q^2[q,q^4;q^{16}]_\infty [q^6;q^{32}]_\infty}{[q^2,q^5;q^{16}]_\infty [q^4,q^{10},q^{16};q^{32}]_\infty}$$

$$+\frac{[q^3,q^4;q^{16}]_\infty}{[q^5;q^{16}]_\infty[q^4,q^{20};q^{32}]_\infty}+\frac{q[q^3,q^8;q^{16}]_\infty[q^2;q^{32}]_\infty}{[q^4,q^5;q^{16}]_\infty[-1;q^{16}]_\infty[q^6,q^{16};q^{32}]_\infty}$$

$$-\frac{q[q^4,q^5;q^{16}]_\infty[q^2;q^{32}]_\infty}{[q^3,q^8;q^{16}]_\infty[-1;q^{16}]_\infty[q^8,q^{10};q^{32}]_\infty}-\frac{q[q^3,q^4;q^{16}]_\infty[q^2,q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty[q^4,q^6,q^8,q^{16};q^{32}]_\infty} \qquad (3.2)$$

Proof ;

Define the power series γ(q) and δ(q) by

$$S(q)=\frac{[q^3;q^8]_\infty}{[q;q^8]_\infty}$$

$$=\frac{[q^3,q^5;q^{16}]_\infty}{[q,q^7;q^{16}]_\infty}=\gamma(q^2)+q\,\delta(q^2) \qquad \text{[by (2.8)]}$$

To prove (3.1), we need to show that

$$\gamma(q^2)=\frac{q^2[q^2,q^4,q^5;q^{16}]_\infty}{[q^6,q^9;q^{16}]_\infty[q^4,q^{16};q^{32}]_\infty}-\frac{q^3[q,q^2,q^4;q^{16}]_\infty[q^{10};q^{32}]_\infty}{[q^3,q^6;q^{16}]_\infty[q^4,q^{14},q^{16};q^{32}]_\infty}$$

$$+\frac{[q^4,q^5,q^7;q^{16}]_\infty}{[q^3,q^9;q^{16}]_\infty[q^4,q^{20};q^{32}]_\infty}$$

$$\delta(q^2)=-\frac{[q^3,q^8;q^{16}]_\infty[q^{10};q^{32}]_\infty}{[q^4,q^5;q^{16}]_\infty[-1;q^{16}]_\infty[q^{14},q^{16};q^{32}]_\infty}$$

$$+\frac{[q^4,q^5;q^{16}]_\infty[q^6;q^{32}]_\infty}{[q^3,q^8;q^{16}]_\infty[-1;q^{16}]_\infty[q^8,q^{14};q^{32}]_\infty}$$

$$+\frac{[q^3,q^4;q^{16}]_\infty[q^{10},q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty[q^4,q^8,q^{14},q^{16};q^{32}]_\infty}$$

From (2.10), we have

$$\frac{[b_1/a_1,b_2/a_1,b_3/a_1,b_4/a_1,b_5/a_1;q]_\infty}{[a_2/a_1,a_3/a_1,a_4/a_1,a_5/a_1;q]_\infty}+\frac{[b_1/a_2,b_2/a_2,b_3/a_2,b_4/a_2,b_5/a_2;q]_\infty}{[a_1/a_2,a_3/a_2,a_4/a_2,a_5/a_2;q]_\infty}$$

$$+\frac{[b_1/a_3,b_2/a_3,b_3/a_3,b_4/a_3,b_5/a_3;q]_\infty}{[a_1/a_3,a_2/a_3,a_4/a_3,a_5/a_3;q]_\infty}+\frac{[b_1/a_4,b_2/a_4,b_3/a_4,b_4/a_4,b_5/a_4;q]_\infty}{[a_1/a_4,a_2/a_4,a_3/a_4,a_5/a_4;q]_\infty}$$

$$+\frac{[b_1/a_5,b_2/a_5,b_3/a_5,b_4/a_5,b_5/a_5;q]_\infty}{[a_1/a_5,a_2/a_5,a_3/a_5,a_4/a_5;q]_\infty}=0 \qquad (3.3)$$

Now, setting (a_1, a_2, a_3, a_4, a_5 ; b_1, b_2, b_3, b_4, b_5) = (1, -1, q, q^7, q^{10} ; q^3, q^5, $-q^2$, q^2, q^6) and taking q^{16} for q in (2.3), we get

$$\frac{[q^3,q^5,-q^2,q^2,q^6;q^{16}]_\infty}{[-1,q,q^7,q^{10};q^{16}]_\infty}+\frac{[-q^3,-q^5,q^2,-q^2,-q^6;q^{16}]_\infty}{[-1,-q,-q^7,-q^{10};q^{16}]_\infty}+\frac{[q^2,q^4,-q,q,q^5;q^{16}]_\infty}{[q^{-1},-q^{-1},q^6,q^9;q^{16}]_\infty}$$
$$+\frac{[q^{-4},q^{-2},-q^{-5},q^{-5},q^{-1};q^{16}]_\infty}{[q^{-7},-q^{-7},q^{-6},q^3;q^{16}]_\infty}+\frac{[q^{-7},q^{-5},-q^{-8},q^{-8},q^{-4};q^{16}]_\infty}{[q^{-10},-q^{-10},q^{-9},q^{-3};q^{16}]_\infty}=0$$

$$\Rightarrow \frac{[q^3,q^5;q^{16}]_\infty[q^4;q^{32}]_\infty[q^6;q^{16}]_\infty[q^{16};q^{32}]_\infty}{2[q,q^7;q^{16}]_\infty[q^6;q^{16}]_\infty}$$
$$+\frac{[-q^3,-q^5;q^{16}]_\infty[q^4;q^{32}]_\infty[-q^6;q^{16}]_\infty[q^{16};q^{32}]_\infty}{2[-q,-q^7;q^{16}]_\infty[-q^6;q^{16}]_\infty}$$
$$+\frac{[q,q^2,q^4,q^5;q^{16}]_\infty[q^2;q^{32}]_\infty}{[q^6,q^9;q^{16}]_\infty(-q^{-1})[q;q^{16}]_\infty(q^{-1})[-q;q^{16}]_\infty[q;q^{16}]_\infty}$$
$$+\frac{(-q^{-4})[q^4;q^{16}]_\infty(-q^{-2})[q^2;q^{16}]_\infty(q^{-5})[-q^5;q^{16}]_\infty(-q^{-5})[q^5;q^{16}]_\infty(-q^{-1})[q;q^{16}]_\infty}{[q^3;q^{16}]_\infty(-q^{-7})[q^7;q^{16}]_\infty(q^{-7})[-q^7;q^{16}]_\infty(-q^{-6})[q^6;q^{16}]_\infty}$$
$$+\frac{(-q^{-7})[q^7;q^{16}]_\infty(-q^{-5})[q^5;q^{16}]_\infty(q^{-8})[-q^8;q^{16}]_\infty(-q^{-8})[q^8;q^{16}]_\infty(-q^{-4})[q^4;q^{16}]_\infty}{(-q^{-10})[q^{10};q^{16}]_\infty(q^{-10})[-q^{10};q^{16}]_\infty(-q^{-9})[q^9;q^{16}]_\infty(-q^{-3})[q^3;q^{16}]_\infty}=0$$

[by (2.3), (2.5) and (2.7)]

$$\Rightarrow \frac{[q^3,q^5;q^{16}]_\infty[q^4;q^{32}]_\infty[q^{16};q^{32}]_\infty}{2[q,q^7;q^{16}]_\infty}+\frac{[-q^3,-q^5;q^{16}]_\infty[q^4;q^{32}]_\infty[q^{16};q^{32}]_\infty}{2[-q,-q^7;q^{16}]_\infty}$$
$$-\frac{q^2[q^2,q^4,q^5;q^{16}]_\infty[q^2;q^{32}]_\infty}{[q^6,q^9;q^{16}]_\infty[q^2;q^{32}]_\infty}+\frac{[q,q^2,q^4;q^{16}]_\infty[q^{10};q^{32}]_\infty}{q^{-3}[q^3,q^6;q^{16}]_\infty[q^{14};q^{32}]_\infty}$$
$$-\frac{[q^4,q^5,q^7;q^{16}]_\infty[q^{16};q^{32}]_\infty}{[q^3,q^9;q^{16}]_\infty[q^{20};q^{32}]_\infty}=0 \qquad \text{[by (2.5)]}$$

$$\Rightarrow \frac{[q^3,q^5;q^{16}]_\infty}{[q,q^7;q^{16}]_\infty}+\frac{[-q^3,-q^5;q^{16}]_\infty}{[-q,-q^7;q^{16}]_\infty}$$
$$=\frac{2q^2[q^2,q^4,q^5;q^{16}]_\infty}{[q^6,q^9;q^{16}]_\infty[q^4,q^{16};q^{32}]_\infty}-\frac{2q^3[q,q^2,q^4;q^{16}]_\infty[q^{10};q^{32}]_\infty}{[q^3,q^6;q^{16}]_\infty[q^{14};q^{32}]_\infty[q^4,q^{16};q^{32}]_\infty}$$
$$+\frac{2[q^4,q^5,q^7;q^{16}]_\infty[q^{16};q^{32}]_\infty}{[q^3,q^9;q^{16}]_\infty[q^{20};q^{32}]_\infty[q^4,q^{16};q^{32}]_\infty} \qquad (3.4)$$

$$\therefore \gamma(q^2)=\frac{1}{2}\{S(q)+S(-q)\}$$
$$=\frac{1}{2}\left\{\frac{[q^3,q^5;q^{16}]_\infty}{[q,q^7;q^{16}]_\infty}+\frac{[-q^3,-q^5;q^{16}]_\infty}{[-q,-q^7;q^{16}]_\infty}\right\}$$

$$\Rightarrow \gamma(q^2) = \frac{q^2[q^2,q^4,q^5;q^{16}]_\infty}{[q^6,q^9;q^{16}]_\infty[q^4,q^{16};q^{32}]_\infty} - \frac{q^3[q,q^2,q^4;q^{16}]_\infty[q^{10};q^{32}]_\infty}{[q^3,q^6;q^{16}]_\infty[q^4,q^{14},q^{16};q^{32}]_\infty} + \frac{[q^4,q^5,q^7;q^{16}]_\infty}{[q^3,q^9;q^{16}]_\infty[q^4,q^{20};q^{32}]_\infty} \quad (3.5)$$

[by (3.4)]

Now, setting (a_1, a_2, a_3, a_4, a_5 ; b_1, b_2, b_3, b_4, b_5) = (1, -1, q^6, $-q^6$, q^2 ; q^3, q^5, $-q$,$-q^7$, q^{-2}) and taking q^{16} for q in (3.3), we get

$$\frac{[q^3,q^5,-q,-q^7,q^{-2};q^{16}]_\infty}{[-1,q^6,-q^6,q^2;q^{16}]_\infty} + \frac{[-q^3,-q^5,q,q^7,-q^{-2};q^{16}]_\infty}{[-1,-q^6,q^6,-q^2;q^{16}]_\infty} + \frac{[q^{-3},q^{-1},-q^{-5},-q,q^{-8};q^{16}]_\infty}{[q^{-6},-q^{-6},-1,q^{-4};q^{16}]_\infty} + \frac{[-q^{-3},-q^{-1},q^{-5},q,-q^{-8};q^{16}]_\infty}{[-q^{-6},q^{-6},-1,-q^{-4};q^{16}]_\infty} + \frac{[q,q^3,-q^{-1},-q^5,q^{-4};q^{16}]_\infty}{[q^{-2},-q^{-2},q^4,-q^4;q^{16}]_\infty} = 0$$

$$\Rightarrow \frac{[q^3,q^5,-q,-q^7;q^{16}]_\infty(-q^{-2})[q^2;q^{16}]_\infty[q^{16};q^{32}]_\infty}{2[q^{12};q^{32}]_\infty[q^2;q^{16}]_\infty} + \frac{[-q^3,-q^5,q,q^7;q^{16}]_\infty(q^{-2})[-q^2;q^{16}]_\infty[q^{16};q^{32}]_\infty}{2[q^{12};q^{32}]_\infty[-q^2;q^{16}]_\infty}$$
$$+ \frac{(-q^{-3})[q^3;q^{16}]_\infty(-q^{-1})[q;q^{16}]_\infty(q^{-5})[-q^5;q^{16}]_\infty[q^2;q^{32}]_\infty(-q^{-8})[q^8;q^{16}]_\infty}{(-q^{-6})[q^6;q^{16}]_\infty(q^{-6})[-q^6;q^{16}]_\infty[-1;q^{16}]_\infty(-q^{-4})[q^4;q^{16}]_\infty[q;q^{16}]_\infty}$$
$$+ \frac{(q^{-3})[-q^3;q^{16}]_\infty(q^{-1})[-q;q^{16}]_\infty(-q^{-5})[q^5;q^{16}]_\infty[q;q^{16}]_\infty(q^{-8})[-q^8;q^{16}]_\infty}{(q^{-6})[-q^6;q^{16}]_\infty(-q^{-6})[q^6;q^{16}]_\infty[-1;q^{16}]_\infty(q^{-4})[-q^4;q^{16}]_\infty}$$
$$+ \frac{[q,q^3;q^{16}]_\infty(q^{-1})[-q;q^{16}]_\infty[q^{10};q^{32}]_\infty(-q^{-4})[q^4;q^{16}]_\infty}{(-q^{-2})[q^2;q^{16}]_\infty(q^{-2})[-q^2;q^{16}]_\infty[q^8;q^{32}]_\infty[q^5;q^{16}]_\infty} = 0$$

[by (2.3), (2.5) and (2.7)]

$$\Rightarrow \frac{[q^3,q^5,-q,-q^7;q^{16}]_\infty(-q^{-2})[q^{16};q^{32}]_\infty}{2[q^{12};q^{32}]_\infty} + \frac{[-q^3,-q^5,q,q^7;q^{16}]_\infty(q^{-2})[q^{16};q^{32}]_\infty}{2[q^{12};q^{32}]_\infty}$$
$$- \frac{q^{-1}[q,q^3;q^{16}]_\infty[q^8;q^{16}]_\infty[q^{10};q^{32}]_\infty[q^2;q^{32}]_\infty}{[q^{12};q^{32}]_\infty[q,q^4;q^{16}]_\infty[-1;q^{16}]_\infty[q^5;q^{16}]_\infty}$$
$$+ \frac{q^{-1}[q^6;q^{32}]_\infty[q^2;q^{32}]_\infty[q,q^5;q^{16}]_\infty[q^{16};q^{32}]_\infty[q^4;q^{16}]_\infty}{[q^3;q^{16}]_\infty[q;q^{16}]_\infty[q^{12};q^{32}]_\infty[-1;q^{16}]_\infty[q^8;q^{32}]_\infty[q^8;q^{16}]_\infty}$$
$$+ \frac{q^{-1}[q,q^3,q^4;q^{16}]_\infty[q^2;q^{32}]_\infty[q^{10};q^{32}]_\infty}{[q^4;q^{32}]_\infty[q^8;q^{32}]_\infty[q^5;q^{16}]_\infty[q;q^{16}]_\infty} = 0 \qquad \text{[by (2.5)]}$$

$$\Rightarrow [q^3,q^5,-q,-q^7;q^{16}]_\infty - [-q^3,-q^5,q,q^7;q^{16}]_\infty$$

$$+\ \frac{2[q,q^3;q^{16}]_\infty[q^8;q^{16}]_\infty[q^2,q^{10};q^{32}]_\infty[q^{12};q^{32}]_\infty}{q^{-1}[q,q^4,q^5;q^{16}]_\infty[-1;q^{16}]_\infty[q^{12};q^{32}]_\infty[q^{16};q^{32}]_\infty}$$
$$-\ \frac{2[q,q^4,q^5;q^{16}]_\infty[q^2,q^6;q^{32}]_\infty[q^{16};q^{32}]_\infty[q^{12};q^{32}]_\infty}{q^{-1}[q,q^3;q^{16}]_\infty[q^8;q^{16}]_\infty[-1;q^{16}]_\infty[q^8,q^{12};q^{32}]_\infty[q^{16};q^{32}]_\infty}$$
$$-\ \frac{2q[q^3,q^4;q^{16}]_\infty[q^2,q^{10};q^{32}]_\infty[q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty[q^4,q^8;q^{32}]_\infty[q^{16};q^{32}]_\infty} = 0$$

$$\Rightarrow [q^3,q^5,-q,-q^7;q^{16}]_\infty - [-q^3,-q^5,q,q^7;q^{16}]_\infty$$
$$= \frac{-2q[q^3;q^{16}]_\infty[q^8;q^{16}]_\infty[q^2,q^{10};q^{32}]_\infty}{[q^4,q^5;q^{16}]_\infty[-1;q^{16}]_\infty[q^{16};q^{32}]_\infty}$$
$$+\ \frac{2q[q^4,q^5;q^{16}]_\infty[q^2,q^6;q^{32}]_\infty}{[q^3;q^{16}]_\infty[q^8;q^{16}]_\infty[-1;q^{16}]_\infty[q^8;q^{32}]_\infty}$$
$$+\ \frac{2q[q^3,q^4;q^{16}]_\infty[q^2,q^{10},q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty[q^4,q^8,q^{16};q^{32}]_\infty} \tag{3.6}$$

$$\delta(q^2) = \frac{1}{2q}\{S(q)-S(-q)\}$$
$$= \frac{1}{2q}\left\{\frac{[q^3,q^5;q^{16}]_\infty}{[q,q^7;q^{16}]_\infty} - \frac{[-q^3,-q^5;q^{16}]_\infty}{[-q,-q^7;q^{16}]_\infty}\right\}$$
$$= \frac{1}{2q[q^2,q^{14};q^{32}]_\infty}\left\{[q^3,q^5,-q,-q^7;q^{16}]_\infty - [-q^3,-q^5,q,q^7;q^{16}]_\infty\right\}$$
$$= -\ \frac{[q^3;q^{16}]_\infty[q^8;q^{16}]_\infty[q^{10};q^{32}]_\infty}{[q^4,q^5;q^{16}]_\infty[-1;q^{16}]_\infty[q^{14},q^{16};q^{32}]_\infty}$$
$$+\ \frac{[q^4,q^5;q^{16}]_\infty[q^2,q^6;q^{32}]_\infty}{[q^3;q^{16}]_\infty[q^8;q^{16}]_\infty[-1;q^{16}]_\infty[q^2,q^8,q^{14};q^{32}]_\infty}$$
$$+\ \frac{[q^3,q^4;q^{16}]_\infty[q^{10},q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty[q^4,q^8,q^{14},q^{16};q^{32}]_\infty} \tag{3.7}$$

[by (3.6)]

$\therefore$ S(q) = $\gamma(q^2) + q\,\delta(q^2)$

$$= \frac{q^2[q^2,q^4,q^5;q^{16}]_\infty}{[q^6,q^9;q^{16}]_\infty[q^4,q^{16};q^{32}]_\infty} - \frac{q^3[q,q^2,q^4;q^{16}]_\infty[q^{10};q^{32}]_\infty}{[q^3,q^6;q^{16}]_\infty[q^4,q^{14},q^{16};q^{32}]_\infty}$$
$$+\ \frac{[q^4,q^5,q^7;q^{16}]_\infty}{[q^3,q^9;q^{16}]_\infty[q^4,q^{20};q^{32}]_\infty} - \frac{q[q^3,q^8;q^{16}]_\infty[q^{10};q^{32}]_\infty}{[q^4,q^5;q^{16}]_\infty[-1;q^{16}]_\infty[q^{14},q^{16};q^{32}]_\infty}$$
$$+\ \frac{q[q^4,q^5;q^{16}]_\infty[q^6;q^{32}]_\infty}{[q^3,q^8;q^{16}]_\infty[-1;q^{16}]_\infty[q^8,q^{14};q^{32}]_\infty}$$

$$+\frac{q[q^3,q^4;q^{16}]_\infty[q^{10},q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty[q^4,q^8,q^{14},q^{16};q^{32}]_\infty}$$

Now,

$$S(q)^{-1} = \frac{[q,q^7;q^{16}]_\infty}{[q^3,q^5;q^{16}]_\infty} \qquad \text{[by (2.9)]}$$

$$= \gamma_1(q^2)+q\delta_1(q^2)$$

$$\therefore \gamma_1(q^2)=\frac{1}{2}\left\{S(q)^{-1}+S(-q)^{-1}\right\}$$

$$=\frac{1}{2}\left\{\frac{[q,q^7;q^{16}]_\infty}{[q^3,q^5;q^{16}]_\infty}+\frac{[-q,-q^7;q^{16}]_\infty}{[-q^3,-q^5;q^{16}]_\infty}\right\} \tag{3.8}$$

[by (2.5)]

Now, setting $(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3, b_4, b_5) = (1, -1, q^3, q^5, q^{10}; q, q^7, -q^2, q^2, q^6)$ and taking q^{16} for q in (3.3), we get

$$\frac{[q,q^7,-q^2,q^2,q^6;q^{16}]_\infty}{[-1,q^3,q^5,q^{10};q^{16}]_\infty}+\frac{[-q,-q^7,q^2,-q^2,-q^6;q^{16}]_\infty}{[-1,-q^3,-q^5,-q^{10};q^{16}]_\infty}+\frac{[q^{-2},q^4,-q^{-1},q^{-1},q^3;q^{16}]_\infty}{[q^{-3},-q^{-3},q^2,q^7;q^{16}]_\infty}$$
$$+\frac{[q^{-4},q^2,-q^{-3},q^{-3},q\ ;q^{16}]_\infty}{[q^{-5},-q^{-5},q^{-2},q^5;q^{16}]_\infty}+\frac{[q^{-9},q^{-3},-q^{-8},q^{-8},q^{-4};q^{16}]_\infty}{[q^{-10},-q^{-10},q^{-7},q^{-5};q^{16}]_\infty}=0$$

$$\Rightarrow\frac{[q,q^7;q^{16}]_\infty[q^4;q^{32}]_\infty[q^6;q^{16}]_\infty[q^{16};q^{32}]_\infty}{2[q^3,q^5;q^{16}]_\infty[q^6;q^{16}]_\infty}$$
$$+\frac{[-q,-q^7;q^{16}]_\infty[q^4;q^{32}]_\infty[-q^6;q^{16}]_\infty[q^{16};q^{32}]_\infty}{2[-q^3,-q^5;q^{16}]_\infty[-q^6;q^{16}]_\infty}$$
$$+\frac{(-q^{-2})[q^2;q^{16}]_\infty[q^3,q^4;q^{16}]_\infty(q^{-1})[-q;q^{16}]_\infty(-q^{-1})[q;q^{16}]_\infty}{(-q^{-3})[q^3;q^{16}]_\infty(q^{-3})[-q^3;q^{16}]_\infty[q^2,q^7;q^{16}]_\infty}$$
$$+\frac{[q,q^2;q^{16}]_\infty(-q^{-4})[q^4;q^{16}]_\infty(q^{-3})[-q^3;q^{16}]_\infty(-q^{-3})[q^3;q^{16}]_\infty}{[q^5;q^{16}]_\infty(-q^{-5})[q^5;q^{16}]_\infty(q^{-5})[-q^5;q^{16}]_\infty(-q^{-2})[q^2;q^{16}]_\infty}$$
$$+\frac{(-q^{-9})[q^9;q^{16}]_\infty(-q^{-3})[q^3;q^{16}]_\infty(q^{-8})[-q^8;q^{16}]_\infty(-q^{-8})[q^8;q^{16}]_\infty(-q^{-4})[q^4;q^{16}]_\infty}{(-q^{-10})[q^{10};q^{16}]_\infty(q^{-10})[-q^{10};q^{16}]_\infty(-q^{-7})[q^7;q^{16}]_\infty(-q^{-5})[q^5;q^{16}]_\infty}=0$$

[by (2.3) and (2.7)]

$$\Rightarrow\frac{[q,q^7;q^{16}]_\infty[q^4,q^{16};q^{32}]_\infty}{2[q^3,q^5;q^{16}]_\infty}+\frac{[-q,-q^7;q^{16}]_\infty[q^4,q^{16};q^{32}]_\infty}{2[-q^3,-q^5;q^{16}]_\infty}$$

$$- \frac{[q^2,q^3,q^4;q^{16}]_\infty [q^2;q^{32}]_\infty}{q^{-2}[q^2,q^7;q^{16}]_\infty [q^6;q^{32}]_\infty} + \frac{[q,q^2,q^4;q^{16}]_\infty [q^6;q^{32}]_\infty}{q^{-2}[q^2,q^5;q^{16}]_\infty [q^{10};q^{32}]}$$

$$- \frac{[q^3,q^4,q^9;q^{16}]_\infty [q^{16};q^{32}]_\infty}{[q^5,q^7;q^{16}]_\infty [q^{20};q^{32}]_\infty} = 0 \qquad \text{[by (2.5)]}$$

$$\Rightarrow \frac{[q,q^7;q^{16}]_\infty}{[q^3,q^5;q^{16}]_\infty} + \frac{[-q,-q^7;q^{16}]_\infty}{[-q^3,-q^5;q^{16}]_\infty}$$

$$= \frac{2q^2[q^3,q^4;q^{16}]_\infty [q^2;q^{32}]_\infty}{[q^7;q^{16}]_\infty [q^6;q^{32}]_\infty [q^4,q^{16};q^{32}]_\infty} - \frac{2q^2[q,q^4;q^{16}]_\infty [q^6;q^{32}]_\infty}{[q^2,q^5;q^{16}]_\infty [q^{10};q^{32}]_\infty [q^4,q^{16};q^{32}]_\infty}$$

$$+ \frac{2[q^3,q^4;q^{16}]_\infty [q^{16};q^{32}]_\infty}{[q^5;q^{16}]_\infty [q^{20};q^{32}]_\infty [q^4,q^{16};q^{32}]_\infty} \tag{3.9}$$

$$\therefore \gamma_1(q^2) = \frac{q^2[q^3,q^4;q^{16}]_\infty [q^2;q^{32}]_\infty}{[q^7;q^{16}]_\infty [q^4,q^6,q^{16};q^{32}]_\infty} - \frac{q^2[q,q^4;q^{16}]_\infty [q^6;q^{32}]_\infty}{[q^2,q^5;q^{16}]_\infty [q^4,q^{10},q^{16};q^{32}]_\infty}$$

$$+ \frac{[q^3,q^4;q^{16}]_\infty}{[q^5;q^{16}]_\infty [q^4,q^{20};q^{32}]_\infty} \tag{3.10}$$

[by (3.9)]

Again, setting $(a_1, a_2, a_3, a_4, a_5 ; b_1, b_2, b_3, b_4, b_5) = (-1, 1, q^6, -q^6, q^2 ; q^3, q^5, -q, -q^7, q^{-2})$ and taking q^{16} for q in (3.3), we get

$$\frac{[-q^3,-q^5,q,q^7,-q^{-2};q^{16}]_\infty}{[-1,-q^6,q^6,-q^2;q^{16}]_\infty} + \frac{[q^3,q^5,-q,-q^7,q^{-2};q^{16}]_\infty}{[-1,q^6,-q^6,q^2;q^{16}]_\infty} + \frac{[q^{-3},q^{-1},-q^{-5},-q,q^{-8};q^{16}]_\infty}{[-q^{-6},q^{-6},-1,q^{-4};q^{16}]_\infty}$$

$$+ \frac{[-q^{-3},-q^{-1},q^{-5},q,-q^{-8};q^{16}]_\infty}{[q^{-6},-q^{-6},-1,-q^{-4};q^{16}]_\infty} + \frac{[q,q^3,-q^{-1},-q^5,q^{-4};q^{16}]_\infty}{[-q^{-2},q^{-2},q^4,-q^{-4};q^{16}]_\infty} = 0$$

$$\Rightarrow \frac{[-q^3,-q^5,q,q^7;q^{16}]_\infty (q^{-2})[-q^2;q^{16}]_\infty [q^{16};q^{32}]_\infty}{2[q^{12};q^{32}]_\infty [-q^2;q^{16}]_\infty}$$

$$+ \frac{[q^3,q^5,-q,-q^7;q^{16}]_\infty (-q^{-2})[q^2;q^{16}]_\infty [q^{16};q^{32}]_\infty}{2[q^{12};q^{32}]_\infty [q^2;q^{16}]_\infty}$$

$$+ \frac{(-q^{-3})[q^3;q^{16}]_\infty (-q^{-1})[q;q^{16}]_\infty (q^{-5})[-q^5;q^{16}]_\infty [-q;q^{16}]_\infty (-q^{-8})[q^8;q^{16}]_\infty}{(q^{-6})[-q^6;q^{16}]_\infty (-q^{-6})[q^6;q^{16}]_\infty [-1;q^{16}]_\infty (-q^{-4})[q^4;q^{16}]_\infty}$$

$$+ \frac{(q^{-3})[-q^3;q^{16}]_\infty (q^{-1})[-q;q^{16}]_\infty (-q^{-5})[q^5;q^{16}]_\infty [q;q^{16}]_\infty (q^{-8})[-q^8;q^{16}]_\infty}{(-q^{-6})[q^6;q^{16}]_\infty (q^{-6})[-q^6;q^{16}]_\infty [-1;q^{16}]_\infty (q^{-4})[-q^4;q^{16}]_\infty}$$

$$+ \frac{[q,q^3;q^{16}]_\infty (q^{-1})[-q;q^{16}]_\infty [q^{10};q^{32}]_\infty (-q^{-4})[q^4;q^{16}]_\infty [q^4;q^{16}]_\infty}{(q^{-2})[-q^2;q^{16}]_\infty (-q^{-2})[q^2;q^{16}]_\infty [q^4;q^{16}]_\infty [q^8;q^{32}]_\infty [q^5;q^{16}]_\infty} = 0$$

[by (2.3), (2.5) and (2.7)]

$$\Rightarrow \frac{[-q^3,-q^5,q,q^7;q^{16}]_\infty (q^{-2})[q^{16};q^{32}]_\infty}{2[q^{12};q^{32}]_\infty} - \frac{[q^3,q^5,-q,-q^7;q^{16}]_\infty (q^{-2})[q^{16};q^{32}]_\infty}{2[q^{12};q^{32}]_\infty}$$
$$- \frac{(q^{-1})[q^3,q^8;q^{16}]_\infty [q^2;q^{32}]_\infty [q^{10};q^{32}]_\infty}{[q^{12};q^{32}]_\infty [-1;q^{16}]_\infty [q^4;q^{16}]_\infty [q^5;q^{16}]_\infty}$$
$$+ \frac{(q^{-1})[q^6;q^{32}]_\infty [q^2;q^{32}]_\infty [q^5;q^{16}]_\infty [q^{16};q^{32}]_\infty [q^4;q^{16}]_\infty}{[q^3;q^{16}]_\infty [q^{12};q^{32}]_\infty [-1;q^{16}]_\infty [q^8;q^{32}]_\infty [q^8;q^{16}]_\infty}$$
$$+ \frac{(q^{-1})[q,q^3;q^{16}]_\infty [q^2;q^{32}]_\infty [q^{10};q^{32}]_\infty [q^4;q^{16}]_\infty [q^4;q^{16}]_\infty}{[q^4;q^{32}]_\infty [q^4;q^{16}]_\infty [q^5;q^{16}]_\infty [q^8;q^{32}]_\infty [q;q^{16}]_\infty} = 0$$

[by (2.5)]

$$\Rightarrow [-q^3,-q^5,q,q^7;q^{16}]_\infty - [q^3,q^5,-q,-q^7;q^{16}]_\infty$$
$$- \frac{2[q^3,q^8;q^{16}]_\infty [q^2,q^{10};q^{32}]_\infty [q^{12};q^{32}]_\infty}{(q^{-1})[q^4,q^5;q^{16}]_\infty [-1;q^{16}]_\infty [q^{12};q^{32}]_\infty [q^{16};q^{32}]_\infty}$$
$$+ \frac{2[q^4,q^5;q^{16}]_\infty [q^2,q^6,q^{16};q^{32}]_\infty [q^{12};q^{32}]_\infty}{q^{-1}[q^3,q^8;q^{16}]_\infty [-1;q^{16}]_\infty [q^{12};q^{32}]_\infty [q^8;q^{32}]_\infty [q^{16};q^{32}]_\infty}$$
$$+ \frac{2[q^3;q^{16}]_\infty [q^4;q^{16}]_\infty [q^2,q^{10};q^{32}]_\infty [q^{12};q^{32}]_\infty}{(q^{-1})[q^5;q^{16}]_\infty [q^4,q^8;q^{32}]_\infty [q^{16};q^{32}]_\infty} = 0$$

$$\Rightarrow [-q^3,-q^5,q,q^7;q^{16}]_\infty - [q^3,q^5,-q,-q^7;q^{16}]_\infty$$
$$= \frac{2q[q^3,q^8;q^{16}]_\infty [q^2,q^{10};q^{32}]_\infty}{[q^4,q^5;q^{16}]_\infty [-1;q^{16}]_\infty [q^{16};q^{32}]_\infty} - \frac{2q[q^4,q^5;q^{16}]_\infty [q^2,q^6;q^{32}]_\infty}{[q^3,q^8;q^{16}]_\infty [-1;q^{16}]_\infty [q^8;q^{32}]_\infty}$$
$$- \frac{2q[q^3,q^4;q^{16}]_\infty [q^2,q^{10},q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty [q^4,q^8,q^{16};q^{32}]_\infty} \tag{3.11}$$

$$\therefore \delta_1(q^2) = \frac{1}{2q}\left\{S(q)^{-1} - S(-q)^{-1}\right\}$$
$$= \frac{1}{2q}\left\{\frac{[q,q^7;q^{16}]_\infty}{[q^3,q^5;q^{16}]_\infty} - \frac{[-q,-q^7;q^{16}]_\infty}{[-q^3,-q^5;q^{16}]_\infty}\right\}$$
$$= \frac{1}{2q[q^6,q^{10};q^{32}]_\infty}\left\{[-q^3,-q^5,q,q^7;q^{16}]_\infty - [q^3,q^5,-q,-q^7;q^{16}]_\infty\right\}$$
$$= \frac{[q^3,q^8;q^{16}]_\infty [q^2;q^{32}]_\infty}{[q^4,q^5;q^{16}]_\infty [-1;q^{16}]_\infty [q^6,q^{16};q^{32}]_\infty} - \frac{[q^4,q^5;q^{16}]_\infty [q^2;q^{32}]_\infty}{[q^3,q^8;q^{16}]_\infty [-1;q^{16}]_\infty [q^8,q^{10};q^{32}]_\infty}$$
$$- \frac{[q^3,q^4;q^{16}]_\infty [q^2,q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty [q^4,q^6,q^8,q^{16};q^{32}]_\infty} \tag{3.12}$$

[by (3.11)]

$$S(q)^{-1} = \gamma_1(q^2) + q\delta_1(q^2)$$

$$= \frac{q^2[q^3,q^4;q^{16}]_\infty[q^2;q^{32}]_\infty}{[q^7;q^{16}]_\infty[q^4,q^6,q^{16};q^{32}]_\infty} - \frac{q^2[q,q^4;q^{16}]_\infty[q^6;q^{32}]_\infty}{[q^2,q^5;q^{16}]_\infty[q^4,q^{10},q^{16};q^{32}]_\infty}$$

$$+ \frac{[q^3,q^4;q^{16}]_\infty}{[q^5;q^{16}]_\infty[q^4,q^{20};q^{32}]_\infty} + \frac{q[q^3,q^8;q^{16}]_\infty[q^2;q^{32}]_\infty}{[q^4,q^5;q^{16}]_\infty[-1;q^{16}]_\infty[q^6,q^{16};q^{32}]_\infty}$$

$$- \frac{q[q^4,q^5;q^{16}]_\infty[q^2;q^{32}]_\infty}{[q^3,q^8;q^{16}]_\infty[-1;q^{16}]_\infty[q^8,q^{10};q^{32}]_\infty} - \frac{q[q^3,q^4;q^{16}]_\infty[q^2,q^{12};q^{32}]_\infty}{[q^5;q^{16}]_\infty[q^4,q^6,q^8,q^{16};q^{32}]_\infty}$$

[by (2.3) and (2.5)]

[IV] CONCLUSION

In this paper, we have established the infinite product expansion of Ramanujan-Gollnitz-Gordon's continued fraction. In the similar way, we can establish the infinite product expansion of other continued fractions.

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