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VARIATIONAL ITERATION METHOD FOR DISPERSION PHENOMENA ARISING IN LONGITUDINAL DISPERSION OF MISCIBLE FLUID FLOW THROUGH POROUS MEDIA

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ABSTRACT

The present paper discusses the approximate solution of the Burger's equation for longitudinal dispersion of miscible fluid flow through porous media which is obtained by Variation Iteration Method (VIM) using Mathematica 8.0.

Keywords: Variational Iteration Method, Longitudinal Dispersion, Burger's equation, Laminar Flow, Homogeneous Porous Media.

[1] INTRODUCTION

In this paper the Numerical Solution and graphical illustration of the dispersion problem is presented by means of Variational Iteration Method and Mathematica 8.0. This problem has been solved by R. Meher using Adomain decomposition method [3].A clear advantage of VIM over the decomposition method is that no calculation of Adomian's polynomials is needed. Dispersion is the term used to describe the spread of a tracer within in a fluid flowing in a porous medium. Dispersion involves two basic transport phenomena: The convection and molecular diffusion. The convective component refers to the spread of tracer due to variations in velocity of the fluid, the variation in the length of streamlines and the branching nature of the streamlines as they divide and rejoin around the solid particles of the porous medium Molecular diffusion provides the mechanism for tracer particles to transfer from one streamline to another, as well as causing spread along the streamlines.

The Phenomenon of Dispersion refers to unsteady mixing of two miscible fluids displacing one another in a porous medium. Dispersion in a porous medium is a consequence of the different flow paths and speed which make a range of transit time available to a concentration front or a collection of tracer particles convected across a porous medium [7]. Consider saturated flow through a porous medium and let a portion of the flow domain contain a certain mass of solute. This solute will be referred to as a tracer. The tracer, which is a labeled portion of the same liquid identified by its density, color, etc. Experience shows that as flow takes place the tracer gradually spreads and occupies an even-increasing portion of the flow domain, beyond the region it is expected to occupy according to the average flow alone. This spreading phenomenon is called dispersion in porous media.

One of the earliest observations of these phenomena is reported by Slichter (1905), who uses an electrolyte as a tracer in studying the movement of ground water. Slichter observes that at an observation well downstream of the injection point, the tracer's concentration increases gradually and that even in uniform (average) flow field the tracer advances in the direction of the flow in a pear-like shape that becomes longer and wider as it advances. Dispersion phenomena occur in many problems of ground water flow, in chemical engineering processes, in oil reservoir engineering, etc. In ground water flow we encounter it in (a) The transition zone between salt water and fresh water in coastal aquifers; (b) Secondary recovery techniques in oil reservoirs, where the injected fluid dissolves the reservoir's oil; (c) The movement of fertilizers in the soil and the leaching of salts from the soil in agriculture [1] and Scheidegger, A. E.[8].

[2] MATHEMATICAL FORMULATION OF THE PROBLEM

The problem is to find the concentration as a function of time't' and position 'x' as the two miscible fluid flow through porous media on either sides of the mixed region. The single fluid equation describes the motion of fluid. Here the mixing takes place longitudinally as well as transversely at t=0 and a dot of fluid having C_0 concentration is injected over the phase. The dot moves in the direction of flow as well as perpendicular to the flow. Finally it takes the shape of the ellipse with C_n.

According to Darcy's law the equation of continuity for the mixture in case of incompressible fluids is given by

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \bar{v}) = 0 \tag{1}$$

Where 'ρ' is the density for mixture and $\bar{v}$ is the pore seepage velocity.

The equation of diffusion for a fluid flow through a homogeneous porous medium without increasing or decreasing the dispersing material[6] is given by

$$\frac{\partial C}{\partial t} + \nabla.(C\bar{v}) = \nabla.\left[\rho \bar{D} \nabla \left(\frac{C}{\rho}\right)\right] \tag{2}$$

Where 'C' is the concentration of a fluid in a porous media. $\bar{D}$ is the Coefficient of dispersion with nine components D_{ij}. In a laminar flow for an Incompressible fluid through homogeneous porous medium, density 'ρ' is constant. Then equation (2) becomes,

$$\frac{\partial C}{\partial t} + \bar{v}.\nabla C = \nabla.\left(\bar{D}\nabla C\right) \tag{3}$$

Let us assume that the seepage velocity $\bar{v}$ is along the $x-$axis, then $\bar{v} = u\left(x,t\right)$ and the nonzero components will be $D_{11} \approx D_L = \gamma$ (Coefficient of longitudinal dispersion) and other Components will be zero [6].

Equation (3) becomes,

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \gamma \frac{\partial^2 C}{\partial x^2} \tag{4}$$

where u is the component velocity along $x-$axis which is time dependent as well as concentration along $x-$axis in $x \geq 0$ direction and $D_L > 0$ and it is the cross sectional flow velocity in porous media.

$u = \dfrac{C\left(x,t\right)}{C_0}$, where $x > 0$ and for $C_0 \cong 1$ by Mehta (2006). Equation (4) becomes

$$\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x} - \gamma \frac{\partial^2 C}{\partial x^2} = 0 \tag{5}$$

This is the non linear Burger's equation for longitudinal dispersion of miscible fluid flow through porous media.

The theory that follows is confined to dispersion in unidirectional seepage flow through semi-infinite homogeneous porous media. The seepage flow velocity is assumed unsteady. The dispersion systems to be considered are subject to an input concentration of contaminants C_0. The governing partial differential equation (5) for longitudinal hydrodynamic dispersion with in a semi-infinite non adsorbing porous medium in a unidirectional flow field in which γ is the longitudinal dispersion coefficient, C is the average cross-sectional concentration, u is the

unsteady seepage velocity, x is a coordinate parallel to flow and t is time.
The initial and boundary conditions are,

$$C(x,0) = f(x) = e^{-x}, x \geq 0 \qquad \text{(6-a)}$$

$$C(0,t) = \varepsilon \text{ (where } \varepsilon \cong 1), 0 < t \leq 0.001 \qquad \text{(6-b)}$$

Since Concentration is decreasing with distance x. Therefore for the sake of convenience $f(x)$ is considered as negative exponential function (Mehta (2006)).

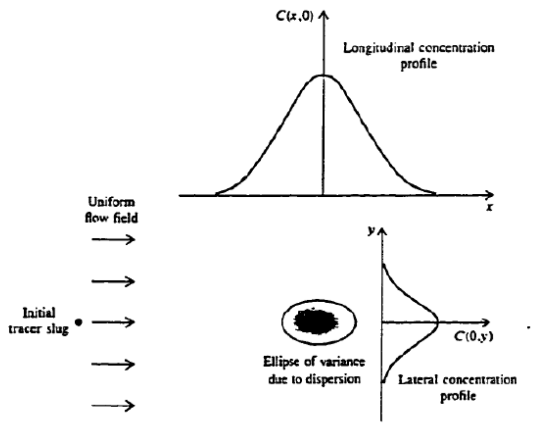


Figure: Dispersion of an instantaneous point source in a uniform flow field [3].

[3] VARIATIONAL ITERATION METHOD

Consider the following differential equation,

$$Lu(t) + Nu(t) = g(t), \quad t > 0 \qquad (7)$$

where the linear operator L is defined as $L = \frac{d^m}{dt^m}$ $m \in N$, N is a nonlinear operator and $g(t)$ is a known analytic function, subject to the initial condition[11],

$$u^k(0) = c_k, \qquad k = 0,1,.......m-1 \qquad (8)$$

Where $c_k{}'s$ are real numbers. According to variational iteration method [2, 9], we construct the correction functional for equation (7) as,

$$u_{k+1}(t) = u_k + \int_0^t \left[\lambda(\tau)\left(Lu_k(\tau) + N\tilde{u}_k(\tau) - g(\tau)\right)\right] d\tau \qquad (9)$$

Where λ is a general Lagrange multiplier, which can be identified optimally via variational theory. We apply restricted variations to nonlinear term Nu so that we can determine the multiplier. Making the above functional stationary, noticing that $\delta\tilde{u}_k = 0$,

$$\delta u_{k+1}(t) = \delta u_k(t) + \delta\int_0^t \left[\lambda(\tau)\left(Lu_k(\tau) - g(\tau)\right)\right] d\tau \qquad (10)$$

Gives the following Lagrange multipliers,

$\lambda = -1$, For $m = 1$

$\lambda = \tau - t$ For $m = 2$,

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \text{ For } m \geq 1 \qquad (11)$$

Therefore, substituting (11) into functional (9), we obtain the following iteration formula,

$$u_{k+1}(t) = u_k(t) + \int_0^t \left[\begin{array}{c} \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \\ \left[\left(Lu_k(\tau) + Nu_k(\tau) - g(\tau)\right)\right] \end{array}\right] d\tau \qquad (12)$$

Now define the operator $A[u]$ (given by Odibat Z M [5] as,

$$A[u] = \int_0^t \left[\begin{array}{c} \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \\ \left[\left(\begin{array}{c} Lu_k(\tau) + \\ Nu_k(\tau) - g(\tau) \end{array}\right)\right] \end{array}\right] d\tau \qquad (13)$$

And define the components $v_k, k = 0,1,2....$, as,

$$\begin{aligned} v_0 &= u_0 \\ v_1 &= A[v_0] \\ v_2 &= A[v_0 + v_1] \\ &. \\ &. \\ v_{k+1} &= A[v_0 + v_1 + ... + v_k] \end{aligned} \qquad (14)$$

Then, consequently, we have

$$u(t) = \lim_{k\to\infty} u_k(t) = \sum_{k=0}^{\infty} v_k \qquad (15)$$

Therefore, as a result, the solution of problem (7) can be derived, using (13) and (14), in the series form,

$$u(t) = \sum_{k=0}^{\infty} v_k(t) \qquad (16)$$

The zeroth (Initial) approximations $v_0 = u_0$ can be freely chosen if it satisfies the initial and boundary conditions of the problem. The success of the method depends on the proper selection of the initial approximation v_0 . However, using the initial values $u^k(0) = c_k,\ k = 0,1,...m-1$ are preferably used for the selective zeroth approximation .In this method the selection of initial approximation v_0 is as follow,

$$v_0 = \sum_{k=0}^{\infty} \frac{c_k}{k!} t^k \qquad (17)$$

[4] SOLUTION

According to variational iteration method, we construct the correction functional for equation (5) as,

$$C_{k+1}(t) = C_k(t) + \int_0^t \left[\lambda(\tau) \left[\frac{\partial C_k}{\partial t} + C_k \frac{\partial C_k}{\partial x} - \frac{\partial^2 C_k}{\partial x^2} \right] \right] d\tau \qquad (18)$$

From the equation (11) we can find the value of Lagrange multiplier and then from equation (13) we define operator $A[C]$ as,

$$A[C] = -\int_0^t \left[\frac{\partial C_k}{\partial t} + C_k \frac{\partial C_k}{\partial x} - \frac{\partial^2 C_k}{\partial x^2} \right] d\tau \qquad (19)$$

Using equation (14) and (19) we get the following.

$$w_1 = -\int_0^t \left[\frac{\partial w_0}{\partial t} + w \frac{\partial w_0}{\partial x} - \frac{\partial^2 w_0}{\partial x^2} \right] d\tau \qquad (20)$$

In this case $w_0 = e^{-x}$. Substituting w_0 into equation (20) and after some simplification, we have

$$w_1(x,t) = (\mathrm{e}^{-2x} + \mathrm{e}^{-x})t \qquad (21)$$

In the same way, we obtained the w_2.

$$w_2(x,t) = \frac{3}{2}\mathrm{e}^{-3x}t^2 + 3\mathrm{e}^{-2x}t^2 + \frac{1}{2}\mathrm{e}^{-x}t^2 + \frac{2}{3}\mathrm{e}^{-4x}t^3 + \mathrm{e}^{-3x}t^3 + \frac{1}{3}\mathrm{e}^{-2x}t^3 \qquad (22)$$

Continuing in this manner, we can obtain the following,

[5] RESULT

5.1 Table

x\t	t=0.0001	t=0.0002	t=0.0003	t=0.0004	t=0.0005	t=0.0006	t=0.0007	t=0.0008	t=0.0009	t=0.001
x=0.1	0.9050	0.9052	0.9054	0.9055	0.9057	0.9059	0.9060	0.9062	0.9064	0.9066
x=0.2	0.8189	0.8190	0.8192	0.8193	0.8195	0.8196	0.8198	0.8199	0.8201	0.8202
x=0.3	0.7409	0.7411	0.7412	0.7413	0.7415	0.7416	0.7417	0.7419	0.7420	0.7421
x=0.4	0.6704	0.6705	0.6707	0.6708	0.6709	0.6710	0.6711	0.6712	0.6713	0.6714
x=0.5	0.6066	0.6067	0.6068	0.6069	0.6070	0.6071	0.6072	0.6073	0.6074	0.6075
x=0.6	0.5489	0.5490	0.5491	0.5492	0.5492	0.5493	0.5494	0.5495	0.5496	0.5497
x=0.7	0.4967	0.4967	0.4968	0.4969	0.4970	0.4970	0.4971	0.4972	0.4973	0.4973
x=0.8	0.4494	0.4495	0.4495	0.4496	0.4497	0.4497	0.4498	0.4499	0.4499	0.4500
x=0.9	0.4066	0.4067	0.4067	0.4068	0.4069	0.4069	0.4070	0.4070	0.4071	0.4071
x=1	0.3679	0.3680	0.3680	0.3681	0.3681	0.3682	0.3682	0.3683	0.3683	0.3684

$$w_3(x,t) = \frac{1}{36}e^{-8x}(6e^{4x}(12+e^x(45+e^x(26+e^x)))t^3 + \frac{3}{2}e^{3x}(65+e^x(196+e^x(123+14e^x)))t^4 + \frac{3}{5}e^{2x}(129+e^x(370+e^x(316+3e^x(22+e^x))))t^5 + e^x(42+e^x(126+e^x(115+3e^x(12+e^x))))t^6 + \frac{4}{7}(1+e^x)(2+e^x)(8+e^x(9+2e^x))t^7) \quad (23)$$

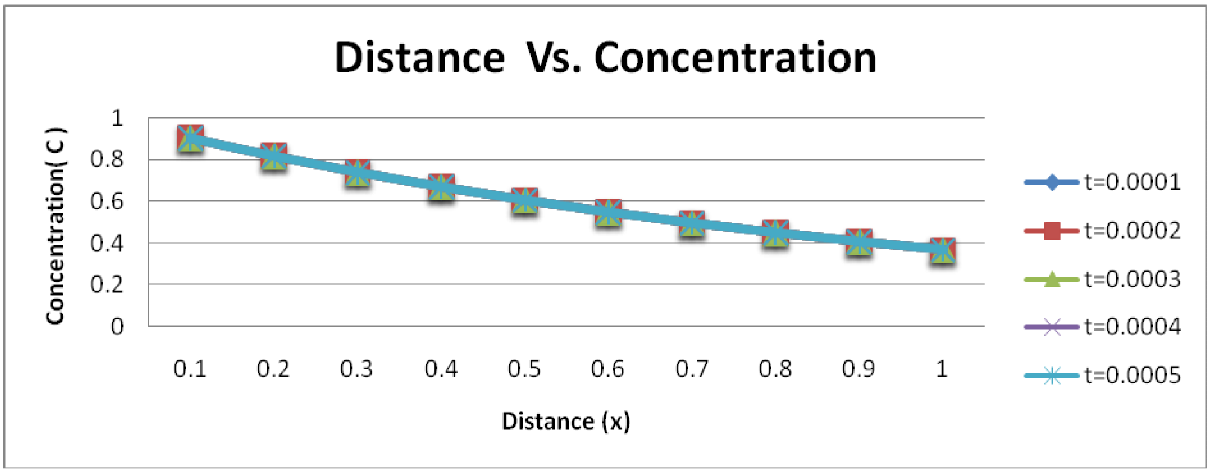
Similarly we can find the later values.

Here we have taken value of $\gamma = 1$ for simplicity and few approximations for numerical purpose.

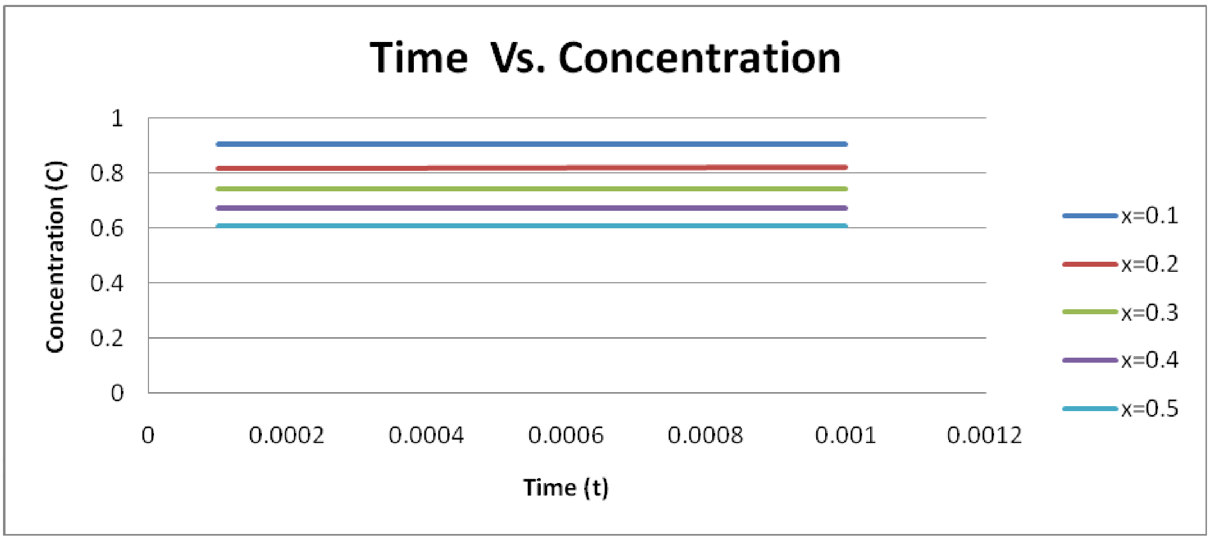
$$C(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + w_4(x,t) \quad (24)$$

5.2 Graphical Representation

1. Distance Vs. Concentration



2. Time Vs. Concentration



[6] CONCLUSION

The solution expresses by equation (24) show that approximate value of concentration (up to five terms) at any x for $0 < t \leq 0.001$. The analytical expressions obtained here are useful to the study of salinity intrusion in groundwater, helpful in making quantitative predictions on the possible contamination of groundwater supplies resulting from groundwater movement through buried wastes [3]. Numerical and graphical representation of solution presents possible concentration of a given dissolved substance in unsteady unidirectional seepage flows through semi-infinite, homogeneous, isotropic porous media subject to the source concentrations that vary negative exponentially with distance. Approximate solution that we have obtained shows Concentration decreases with distance and slightly increases with time.

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