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BOUNDS ON TOTAL DOMINATION IN SQUARES OF GRAPHS

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ABSTRACT:

Let $G^2 = (V, E(G^2))$ be the square of a connected graph G. A dominating set D of G^2 is said to be total dominating set, if for every vertex $v \in V(G^2)$, there exists a vertex $u \in D$, $u \neq v$, such that u is adjacent to v. The total domination number of G^2, denoted by $\gamma_t(G^2)$ is the minimum cardinality of a total dominating set of G^2. In this paper, many bounds on $\gamma_t(G^2)$ were obtained in terms of the elements of G. Also its relationship with other domination parameters were obtained.

Keywords: Graph, Square graph, Dominating set, Total dominating set and Total domination number.

Subject Classification Numbers: AMS – 05C69, 05C70.

[I] INTRODUCTION

In this paper, we follow the notations of [1]. All the graphs considered here are simple, finite, non-trivial, undirected and connected. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively.

In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v.

The minimum(maximum)degree among the vertices of G is denoted by $\delta(G)(\Delta(G))$. A vertex of degree one is called an end vertex. Any vertex of degree greater than one is called an internal vertex. The term $\alpha_0(G)(\alpha_1(G))$ denote the minimum number of vertices(edges) cover of G. Further, $\beta_0(G)(\beta_1(G))$ represents the vertex (edge) independence number of G.

The square of a graph G denoted by G^2, has the same vertices as in G and the two vertices u and v are joined in G^2 if and only if they are joined in G by a path of length one or two. The concept of squares of graphs was introduced in [2].

A set $S \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V - S$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set S is called total dominating set, if for every vertex $v \in V$, there exists a vertex $u \in S$, $u \neq v$ such that u is adjacent to v. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G. A dominating set S is called an independent dominating set, if no two vertices of S are adjacent. The independent domination number of G, denoted by $i(G)$ is the minimum cardinality taken over all independent dominating sets of G.

Further, a dominating set S is called an end dominating set of G, if S contains all the end vertices in G. The minimum cardinality of vertices in such a set is called the end domination number of G and is denoted by $\gamma_e(G)$. Domination related parameters are now well studied in graph theory (see [3], [4]).

A subset S' of E is called an edge dominating set of G if every edge not in S' is adjacent to some edges in S'. The edge domination number of G, denoted by $\gamma'(G)$ is the minimum cardinality taken over all edge dominating sets of G (see [6]).

A set $D \subseteq V(G^2)$ is said to be a dominating set of G^2, if every vertex not in D is adjacent to a vertex in D. The minimum cardinality of vertices in such a set is called the domination number of G^2 and is denoted by $\gamma(G^2)$ (see [5]).

Analogously, a dominating set D of G^2 is said to be total dominating set, if for every vertex $v \in V(G^2)$, there exists a vertex $u \in D$, $u \neq v$, such that u is adjacent to v. The total domination number of G^2, denoted by $\gamma_t(G^2)$ is the minimum cardinality of a total dominating set of G^2. In this paper, many bounds on $\gamma_t(G^2)$ were obtained in terms of elements of G. Also its relationship with other domination parameters were obtained.

[II] RESULTS

Initially, we relate the total domination number of G and G^2.

Theorem 2.1. For any connected graph G, $\gamma_t(G^2) \leq \gamma_t(G)$. Equality holds if and only if $diam(G) \leq 3$.

Proof. For $p = 2$, the result is obvious. Let $p \geq 3$, Suppose $C = \{v_1, v_2, ..., v_i\}$ be the set of all internal vertices in G. Then there exists a minimal set $C' = \{v_1, v_2, ..., v_k\} \subseteq C$ such that $C' \cup H$, where $H \subseteq V - C'$ forms a γ_t-set of G. Now without loss of generality in G^2, suppose $D_1 = \{v_1, v_2, ..., v_n\}$ be the set of minimal vertices with $\deg(v_j) \geq 2, 1 \leq j \leq n$ such that $D_1 \cup I$, where $I \subseteq V(G^2) - D_1$ covers all the vertices of G^2 and the subgraph $\langle D_1 \cup I \rangle$ has no isolated vertex. Clearly, $D_1 \cup I$ forms a minimal total dominating set of G^2. Since distance between vertices of G^2 is at most two, it follows that, $|D_1 \cup I| \leq |C' \cup H|$. Therefore, $\gamma_t(G^2) \leq \gamma_t(G)$.

Suppose $diam(G) \leq 3$, then $|C' \cup H| = 2$

$= |D_1 \cup I|$. Clearly, it follows that $\gamma_t(G^2) = \gamma_t(G)$.

In contradiction to the above, suppose $diam(G) \geq 4$. Then in this case, there exists at least one vertex $v \in V(G)$ such that $|C' \cup H| \geq 3$. Clearly, $|D_1 \cup I| < |C' \cup H|$ and hence $\gamma_t(G^2) < \gamma_t(G)$. □

In the following Theorem, we characterize total domination number of G^2 in terms of diameter of G.

Theorem 2.2. For any graph G, $\gamma_t(G^2) = 2$ if G is

(i) isomorphic to a tree T with $diam(T) \leq 6$

(ii) containing a cycle C with $diam(C) \leq 2$ and $diam(G) \leq 6$.

Proof. (i) For any connected graph G, if $diam(G) \leq 6$ then $\gamma_t(G^2) = 2$ with G containing a cycle C with $diam(C) \leq 2$. Suppose G does not contain any cycle and $diam(G) > 6$, then there exists at least one vertex $v \in V(G^2)$ in the subgraph $\langle D \rangle$ which is isolated, where D is a γ_t-set of G^2. Clearly, $D \cup \{v\}$ forms a minimal γ_t-set of G^2 and $|D \cup \{v\}| \geq 3$ and hence $\gamma_t(G^2) \geq 3$, a contradiction.

Conversely, if $diam(G) \leq 6$, then in G^2, $|D| = 2$ with the subgraph $\langle D \rangle$ has no isolated vertex. Clearly, $\gamma_t(G^2) = 2$.

(ii) Suppose $diam(G) \leq 6$ and assume G contains a cycle C with $diam(C) \geq 3$. Then in G^2, there exists at least one vertex $u \in V(G^2)$ of cycle C which makes the subgraph $\langle D \rangle$ containing an isolate. Clearly, $D \cup \{u\}$ forms a minimal γ_t-set of G^2 and hence $|D \cup \{u\}| \geq 3$, a contradiction to the fact that $diam(C) \geq 3$. Therefore $diam(C) \leq 2$ with $diam(G) \leq 6$. □

The following Corollary is immediate from the above Theorem.

Corollary 2.1. $\gamma_t(K_p^2) = \gamma_t(W_p^2) = \gamma_t(K_{1,n}^2) = \gamma_t(K_{p_1,p_2}^2) = 2$.

The following Theorem relates total domination in G^2 with domination number of G.

Theorem 2.3. For any connected (p,q)-graph G with $p \geq 3$, $\gamma_t(G^2) + \gamma(G) \leq p$. Equality holds for C_3, P_3, C_4, P_4.

Proof. For $p = 2$, $\gamma_t(G^2) + \gamma(G) \not\leq p$. Now we consider the graph with $p \geq 3$ and let $S = \{v_1, v_2, ..., v_n\} \subseteq V(G)$ be the minimal dominating set of G such that $|S| = \gamma(G)$. Now in G^2, since $V(G) = V(G^2)$, let $D \subseteq S$ be a minimal dominating set of G^2. Suppose $V_1 = V(G^2) - D$ and $D' \subseteq V_1$ such that $D' \in N(D)$ in G^2. Then $D \cup D'$ forms a total dominating set in G^2. Clearly, $|D \cup D'| \cup |S| \leq p$ and hence $\gamma_t(G^2) + \gamma(G) \leq p$.

For equality, we have the following cases.

Case 2.3.1: Suppose G is isomorphic to C_3 or P_3. Then in this case, $|D \cup D'| = 2|S|$. Clearly, it follows that, $\gamma_t(G^2) + \gamma(G) = p$.

Case 2.3.2: Suppose G is isomorphic to C_4 or P_4. Then in this case, $|D \cup D'| = 2 = |S|$. Clearly, it follows that, $\gamma_t(G^2) + \gamma(G) = p$. □

The following Theorem relates total domination in G^2 with domination number of G.

Theorem 2.4. For any connected graph G, $\gamma_t\left(G^2\right) \leq \gamma'\left(G\right)+1$.

Proof. Let $F' = \{e_1, e_2, e_3, ..., e_m\}$ be the set of all end edges in G. Suppose $E - F' = I$, then $S' \subseteq I$ forms an γ'-set of G. Further, if $E - F' = \phi$, then there exists at least one edge $\{e\} \in F'$ such that $S' = \{e\}$ forms a minimal edge dominating set of G. Now in G^2, since $V(G) = V(G^2)$, let $D' = \{v_1, v_2, v_3, ..., v_m\} \subseteq V(G^2)$ be the set of vertices which are incident to the edges of S'. Further, $D = \{v_1, v_2, v_3, ..., v_k\} \subseteq D'$ be the minimum set of vertices such that $N[D] = V(G^2)$. Suppose the subgraph $\langle D \rangle$ has no isolate, then D itself is a total dominating set of G^2. Otherwise, there exists at least one vertex $u \notin D$, such that the subgraph $\langle D \cup \{v\} \rangle$ forms a minimal total dominating set of G^2. Clearly, it follows that, $|D \cup \{v\}| \leq |S'| + 1$. Therefore, $\gamma_t\left(G^2\right) \leq \gamma'\left(G\right)+1$. □

In the following Theorem, we give the lower bound on $\gamma_t(G^2)$ in terms of vertices and degree of G.

Theorem 2.5. For any connected (p,q)-graph G, $\gamma_t(G^2) \leq \left\lceil \frac{p}{\Delta(G)} \right\rceil$. Equality holds if $diam(G) \leq 3$.

Proof. Let D be a dominating set of G^2 and $V_1 = V(G^2) - D$ such that $V_1 \in N(D)$. Suppose $D_1 \subseteq V_1$ and $D_1 \in N(D)$, then $D \cup D_1$ forms a minimal total dominating set of G^2. Further, if $C = \{v_1, v_2, ..., v_n\}$ be the set of all non end vertices in G, then there exists at least one vertex v of maximum degree $\Delta(G)$ in C such that $|D \cup D_1| \cdot \Delta(G) \leq p$. Clearly, it follows that $\gamma_t(G^2) \leq \left\lceil \frac{p}{\Delta(G)} \right\rceil$.

For the equality, suppose $diam(G) \leq 3$. For the graphs with $diam(G) \leq 2$, we have $K_{1,n}$, $n \geq 2$ as a spanning subgraph of G. Since $G^2 = K_{1+n}$ and $\Delta(G^2) = n$, then $\gamma_t(G^2) \leq \left\lceil \frac{p}{\Delta(G)} \right\rceil$. Suppose $diam(G) = 3$, then the spanning subgraph is a double star. Let u and v be the corresponding adjacent vertices with maximum degree in double star. Let $N(u) = v_i$ and $N(v) = v_j$, each v_i is adjacent to v and each v_j is adjacent to u. Also $N(u)$ and $N(v)$ together with u and v forms complete sub graphs in its square. The minimal total dominating set of G^2 contains exactly u and v. Then its cardinality is $2 = \gamma_t(G^2) = \left\lceil \frac{p}{\Delta(G)} \right\rceil$. □

The following Theorem relates total domination number of G^2 with independent domination number of G.

Theorem 2.6. For any connected (p,q)-graph G, $\gamma_t(G^2)+i(G)\le p+1$.

Proof. Let $S=\{v_1,v_2,v_3,...,v_k\}\subseteq V(G)$ be the minimum set of vertices such that $d(u,v)\ge 2$, for all $u,v\in S$, which covers all the vertices in G. Clearly, S forms an independent dominating set of G. Now in G^2, let $D_1=\{v_1,v_2,v_3,...,v_n\}\subseteq V(G^2)$ be the minimal set of vertices which covers all the vertices in G^2. If the subgraph $\langle D_1\rangle$ does not contain any isolated vertex, then D_1 itself is a total dominating set of G^2. Otherwise , there exists at least one vertex $u\notin D_1$ makes the subgraph $\langle D_1\cup\{u\}\rangle$ not containing any isolated vertex. Clearly, $D_1\cup\{u\}$ forms a minimal total dominating set of G^2. Therefore, it follows that, $|D_1\cup\{u\}|+|S|\le p+1$. Hence $\gamma_t(G^2)+i(G)$ $\le p+1$. □

In the following Theorems, we give lower bounds on $\gamma_t(G^2)$.

Theorem 2.7. For any connected (p,q)-graph G, $\gamma_t(G^2)\le p-\alpha_0(G)+1$. Equality holds if and only if G is isomorphic to K_p.

Proof. Let $B=\{v_1,v_2,...,v_k\}$ be the minimum set of vertices which covers all the edges in G such that $|B|=\alpha_0(G)$. Now without loss of generality in G^2, suppose $D'=\{v_1,v_2,...,v_n\}$ be the minimal set of vertices such that $N[D']=V(G^2)$ doesnot contain any isolated vertex, then D' itself is a total dominating set of G^2. Otherwise, $D'\cup H=D$ where $H\subseteq V(G^2)-D'$, and the subgraph $\langle D\rangle$ has no isolated vertex forms the minimal total dominating set of G^2. Clearly, it follows that $|D|\le p-|B|+1$. Therefore, $\gamma_t(G^2)\le p-\alpha_0(G)+1$.

Suppose G is not isomorphic to K_p. Then $|D|\ge 2$ or $|B|\le p-1$. Clearly, it follows that $|D|<p-|B|+1$, a contradiction.

Conversely, if G is isomorphic to K_p. Then in this case $|D'|=2$ and $|B|=p-1$. Clearly, it follows that $|D|=p-|B|+1$ and hence $\gamma_t(G^2)=p-\alpha_0(G)+1$. □

Theorem 2.8. For any connected (p,q)-graph G, $\gamma_t(G^2)\le p-\gamma_e(G)+1$. Equality holds if G is isomorphic to $K_{1,n}$.

Proof. To prove this result, we consider the following two cases.

Case 2.8.1: Suppose G has no end vertices. Let $S=\{v_1,v_2,...,v_k\}$ be the minimal set of vertices with $\deg(v_i)\ge 2$, $1\le i\le k$, which covers all the vertices in G. Clearly, S forms an end dominating set of G. Now without loss of generality in G^2, since $V(G)=V(G^2)$, there exists a vertex set $D_1'=\{v_1,v_2,...,v_j\}$ which covers all the vertices in G^2. Suppose the subgraph $\langle D_1'\rangle$ contains no isolated vertex. Then D_1' itself is a γ_t- set of G^2. Otherwise, there exists at least one vertex $u\notin D_1'$ such that $D_1'\cup\{u\}=D$

forms a minimal total dominating set of G^2. Clearly, it follows that $|D| \le p - |S| + 1$ and hence $\gamma_t\left(G^2\right) \le p - \gamma_e(G) + 1$.

Case 2.8.2: Suppose G has end vertices, let $F = \{v_1, v_2, ..., v_m\}$ be the set of all end vertices in G and $V' = V(G) - F$. Then there exists a vertex set $A \subseteq V'(G)$ which are not adjacent to the vertices of F, such that $F \cup A$ forms a minimal end dominating set of G^2. Now in G^2, since $V(G) = V\left(G^2\right)$, let $D \subseteq V'$ forms a minimal total dominating set of G^2. Clearly, it follows that $|D| \le p - |F \cup A| + 1$. Therefore, $\gamma_t\left(G^2\right) \le p - \gamma_e(G) + 1$.

Suppose G is isomorphic to $K_{1,n}$, $1 + n = p$. Then in this case, $|D| = 2$ and F itself is a minimal γ_e- set of G such that $|F| = p - 1$. Therefore, it follows that, $\gamma_t\left(G^2\right) = p - \gamma_e(G) + 1$. □

Theorem 2.9. For any connected (p, q)- graph G, which is not a path, $\gamma_t\left(G^2\right) \le p - diam(G)$. Equality holds if G is isomorphic to C_3 and C_4.

Proof. Suppose G is isomorphic to a path P_p. Then in this case, $p - diam(G) = 1$. Since for any connected graph G, $\gamma_t\left(G^2\right) \ge 2$. Clearly, $\gamma_t\left(G^2\right) \not\le p - diam(G)$. Let G is not isomorphic to a path P_p and $u, v \in V(G)$ be any two distinct vertices such that the distance between u and v forms the $diam(G)$. If $D = \{v_1, v_2, ..., v_n\} \subseteq V(G^2)$ be the minimal set which covers all the vertices of G^2 and if the subgraph $\langle D \rangle$ has no isolates, then D itself is a total dominating set of G^2. Otherwise, there exists a subset $H \subseteq V(G^2) - D$ such that $D \cup H$ forms a minimal total dominating set of G^2. Clearly, it follows that $|D \cup H| \le p - diam(G)$. Therefore, $\gamma_t\left(G^2\right) \le p - diam(G)$.

Suppose G is isomorphic to either C_3 or C_4. Then in this case, $diam(G) = p - 2$ and $\gamma_t\left(G^2\right) = 2$. Clearly, it follows that $\gamma_t\left(G^2\right) = p - diam(G)$. □

Theorem 2.10. For any connected (p, q)- graph G, $\left\lceil \frac{\gamma_t(G^2)}{2} \right\rceil \le p - \Delta(G)$. Equality holds if and only if $\deg(v) = p - 1$, where $v \in V(G)$.

Proof. Suppose $C = \{v_1, v_2, ..., v_n\}$ be the set of all non end vertices in G, then there exists at least one vertex $v \in V(G)$ of maximum degree such that $\deg(v) = \Delta(G)$. Now without loss of generality in G^2, $V(G) = V(G^2)$. Suppose $D \subseteq C$ in G^2, is a minimal γ- set of G^2, then $D \cup H$, where $H \in N(D)$ and $H \subseteq V(G^2) - D$ forms a minimal total dominating set of G^2. Further, since two vertices in G^2 are joined which are at distance at most two, it follows that,

$\left\lceil \frac{|D \cup H|}{2} \right\rceil \leq p - \Delta(G)$. Clearly, $\left\lceil \frac{\gamma_t(G^2)}{2} \right\rceil \leq p - \Delta(G)$.

Suppose G contains a vertex $v \in V(G)$ with $\deg(v) < p-1$. Then in this case, $\Delta(G) < p-1$. Further, since $diam(G) \geq 3$, we have $\gamma_t(G^2) \geq 2$. Clearly, it follows that $\left\lceil \frac{\gamma_t(G^2)}{2} \right\rceil < p - \Delta(G)$, a contradiction.

Conversely, if G contains a vertex $v \in V(G)$ with $\deg(v) = p-1$. Then in this case, $\Delta(G) = p-1$, $diam(G) \leq 2$. Clearly, we have $\gamma_t(G^2) = 2$ and hence $\frac{\gamma_t(G^2)}{2} = p - \Delta(G)$.

□

Theorem 2.11. If every non end vertex of a tree T is adjacent to at least one end vertex, then $\gamma_t(G^2) \leq \left\lceil \frac{p-m}{2} \right\rceil + 1$, where m is the number of end vertices in T.

Proof. Let $F = \{v_1, v_2, ..., v_n\}$ be the set of all end vertices in T such that $|F| = m$. Now in G^2, since $V(T) = V(T^2)$, there exists a set $D \subseteq V(T^2) - F$, which covers all the vertices in T^2. Suppose the subgraph $\langle D \rangle$ has no isolated vertex, then D itself is a minimal total dominating set of T^2. Otherwise, there exists at least one vertex $u \notin D$ such that $D \cup \{u\}$ forms minimal γ_t- set of T^2. Since every tree T contains at least two end vertices, it follows that $|D \cup \{u\}| - 1 \leq \frac{(p - |F|)}{2}$. Therefore, $\gamma_t(T^2) \leq \left\lceil \frac{p-m}{2} \right\rceil + 1$.

Theorem 2.12. For any connected (p,q)-graph G with $p \geq 3$ vertices, $\gamma_t(G^2) \leq \left\lceil \frac{(2q-p)}{3} \right\rceil + 1$. Equality holds if G is isomorphic to P_4, P_5, K_3 or $K_{1,n}$ with $n \leq 4$.

Proof. Let $C = \{v_1, v_2, ..., v_k\}$ be the set of all non end vertices which are adjacent to the support vertices in G. Further let $C' = \{u_1, u_2, ..., u_n\}$ be the set of vertices with $\deg(u_i) \geq 2$ and $u_i \neq v_j$, $1 \leq i \leq n$, $1 \leq j \leq k$. Now without loss of generality in G^2, $D = C \cup C_1'$, where $C_1' \subseteq C'$, covers all the vertices in G^2. Suppose the subgraph $\langle D \rangle$ has no isolate, then D itself is a minimal total dominating set of G^2. Otherwise, there exists at least one vertex $w \in V(G^2)$, such that $D \cup \{w\}$ forms a minimal γ_t- set of G^2. Clearly, it follows that $|D \cup \{w\}| \leq \left\lceil \frac{(2q-p)}{3} \right\rceil + 1$. Therefore, $\gamma_t(G^2) \leq \left\lceil \frac{(2q-p)}{3} \right\rceil + 1$.

For the equality, suppose G is isomorphic to either P_4 or P_5 or K_3 or $K_{1,n}$ with $n \leq 4$. Then in this case, $|D| = 2$ and $(2q-p) \leq 3$. Clearly, it follows that, $\gamma_t(G^2) =$

$\left\lceil \frac{(2q-p)}{3} \right\rceil + 1$.

Finally, we give the Nordhaus – Gaddum type result.

Theorem 2.13. If G and $\bar{G}$ are connected graphs with p - vertices, then

1. $\gamma_t\left(G^2\right)+\gamma_t\left(\bar{G}^2\right)=4=\gamma_t\left(G^2\right)\cdot\gamma_t\left(\bar{G}^2\right)$ for $p \le 3$.
2. $4 \le \gamma_t\left(G^2\right)+\gamma_t\left(\overline{G^2}\right) \le \frac{2p}{3}+2$ and $4 \le \gamma_t\left(G^2\right)\cdot\gamma_t\left(\overline{G^2}\right) \le \frac{4p}{5}$ for $p \ge 4$.

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