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NON LINEAR FOUR - POINT (N - POINT) KRONECKER PRODUCT BOUNDARY VALUE PROBLEM

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ABSTRACT:

In this paper, existence and uniqueness of solutions to four-point and multi-point boundary value problems associated with a system of first order rectangular matrix differential equation involving kronecker products by using variation of parameters formula are derived. These generalize the results obtained for four-point boundary value problems involving kronecker products.

Key Words: Kronecker product, Four-point boundary value problem, Multi-point boundary value problem, Green's Matrix.

AMS (MOS) Subject Classification: 34B15, 34B99.

[I] INTRODUCTION

The study of boundary value problems associated with linear and non-linear differential is an important area of current research and has tremendous applications in various branches of science and technology. In finding solutions to four-point boundary value problems involving, the construction of Green's matrix is vital. It is sufficiently known about the construction of Green's matrix for problems involving non-singular square matrices. The theory for rectangular matrices involves significant difficulties as the inverse of the matrix in the usual sense, does not exist. In this chapter, we establish the solutions of boundary value problems associated with kronecker product system of first order rectangular matrix differential equations. By a suitable transformation, the rectangular matrices are transformed into non-singular square matrices and solutions are finally expressed in terms of the rectangular matrices as in the four- point boundary value problems.

In this paper, we consider the following kronecker product four-point boundary value problem:

$$(P(t)\otimes Q(t))y'(t) + (R(t)\otimes S(t))y(t) = f(t,y(t)),\quad a \le t \le d \qquad (1.1)$$

$$(M_1\otimes N_1)\, y(a + (M_2\otimes N_2)\, y(b) + (M_3\otimes N_3)\, y(c) + (M_4\otimes N_4)\, y(d) = \alpha, \qquad (1.2)$$

where P(t), Q(t), R(t) and S(t) are rectangular matrices of order $(m \times n)$, y(t) is of order $(n^2 \times 1)$, $f : [a, d] \times R^{n^2} \to R^{m^2}$ and the components of P(t), Q(t), R(t), S(t) and f are continuous on [a, d], we assume that $f(t,0) \equiv 0$ for all $t \in [a, d]$ and f satisfies Lipchitz condition on [a,d]. We also assume that the rows of P(t) and Q(t) are linearly independent on [a,d] and the system (1.1) is consistent. M_1, N_1, M_2 ,N_2, M_3 ,N_3, M_4 and N_4 are matrices of order (m×n) and α is a column matrix of order ($m^2 \times 1$).

This paper is organized as follows: In section 2, we develop the general solution of the homogeneous kronecker product system corresponding to (1.1) in terms of a fundamental matrix. We then establish the variation of parameters formula to find the solution of non-homogeneous kronecker product system (1.1).Section 3, presents criteria for the existence and uniqueness of solutions to four-point boundary value problem. We establish the general solution of the four-point kronecker product boundary value problem in terms of an integral representation involving Green's matrix and we also verify the properties of the Green's matrix. In section 4, we establish criteria for the existence and uniqueness of solutions to multi - point boundary value problem. The results obtained in this paper are exemplified at the end of this paper.

[II] GENERAL SOLUTION OF NON-LINEAR KRONECKER PRODUCT SYSTEM

In this section, the general solution of the homogeneous kronecker product system

$$(P(t)\otimes Q(t))y'(t) + (R(t)\otimes S(t))y(t) = 0 \qquad (2.1)$$

is obtained and thereby establish the general solution of the non-linear kronecker product system (1.1) using variation of parameters method. Let, y(t) = $\left(P^T(t)\otimes Q^T(t)\right)$ z(t). Then the transformed equation of (2.1) is of the form

$$\left(P(t)P^T(t)\otimes Q(t)Q^T(t)\right)z'(t) + \left[\left(P(t)P^{T'}(t)\otimes Q(t)Q^{T'}(t)\right) + \left(R(t)P^T(t)\otimes S(t)Q^T(t)\right)\right] z(t)=0.$$

Since $P(t)P^T(t)\otimes Q(t)Q^T(t)$ is non-singular, follows that

$$z'(t) = -\left(P(t)P^T(t)\otimes Q(t)Q^T(t)\right)^{-1} \left[\left(P(t)P^{T'}(t)\otimes Q(t)Q^{T'}(t)\right) + \left(R(t)P^T(t)\otimes S(t)Q^T(t)\right)\right]z(t),$$

i.e. $$z'(t) = -A^{-1}(t)B(t)z(t), \qquad (2.2)$$

where,

$$A(t) = \left(P(t)P^T(t)\otimes Q(t)Q^T(t)\right),$$
$$B(t) = \left[\left(P(t)P^T(t)\otimes Q(t)Q^{T'}(t)\right) + \left(R(t)P^{T'}(t)\otimes S(t)Q^{T'}(t)\right)\right]$$

and $P^T(t), Q^T(t)$ are the transposes of the matrices P(t) and Q(t).

Theorem 2.1 : If the system of equations (2.1) is consistent, then any solution of (2.1) is of the form $\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)k$, where Φ(t) is a fundamental matrix of (2.2) and k is a constant vector of order ($m^2 \times 1$).

Proof : The transformation y(t) = $\left(P^T(t)\otimes Q^T(t)\right)$ z(t) transforms (2.1) into (2.2). Since Φ(t) is a fundamental matrix of (2.2) it follows that any solution z(t) is of the form z(t) = Φ(t) k, where k is a constant vector of order ($m^2 \times 1$).

Hence, y(t) = $\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)$ k.

Theorem 2.2 : A particular solution $\overline{y}(t)$ of (1.1), is of the form

$$\overline{y}(t)=\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)\int_a^t \Phi^{-1}(s) \left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s, y(s))\, ds.$$

Proof : By the transformation y(t) = $\left(P^T(t)\otimes Q^T(t)\right)$ z(t) transforms the equation (1.1) into

$$z'(t) + A^{-1}(t)B(t)z(t) = A^{-1}(t)\, f\left(t,\left(P^T(t)\otimes Q^T(t)\right)z(t)\right). \qquad (2.3)$$

Now we seek a particular solution of (2.3) in the form $\overline{z}(t)$ = Φ(t) K(t). Then

$$\Phi'(t)K(t) + \Phi(t)K'(t) + A^{-1}(t)B(t)\Phi(t)K(t) = A^{-1}(t)\, f\left(t,\left(P^T(t)\otimes Q^T(t)\right)z(t)\right).$$

$$\Leftrightarrow \Phi(t)K'(t) = A^{-1}(t)\, f\left(t,\left(P^T(t)\otimes Q^T(t)\right)z(t)\right)$$

$$\Leftrightarrow K'(t) = \Phi^{-1}(t)\ A^{-1}(t)\ f\left(t,\left(P^T(t)\otimes Q^T(t)\right)z(t)\right)\ K(t)$$

$$= \int_a^t \Phi^{-1}(s)\ \left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}$$

$$f\left(s,\left(P^T(s)\otimes Q^T(s)\right)z(s)\right)\ ds.$$

Hence , a particular solution of (2.3) is given by

$$\bar{z}(t) = \Phi(t)\int_a^t \Phi^{-1}(s)$$

$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}$$

$$f\left(s,\left(P^T(s)\otimes Q^T(s)\right)z(s)\right) ds.$$

hence a particular solution of (1.1) is of the form $\bar{y}(t)=\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)\int_a^t \Phi^{-1}(s)$

$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}(s)\ f(s, y(s))\ ds.$$

Theorem 2.3 : Any solution of (1.1) is of the form $y(t) = \left(P^T(t)\otimes Q^T(t)\right)\Phi(t)k + \bar{y}(t)$,

where $\bar{y}(t)$ is a particular solution of (1.1).

Proof : It can easily be verified that $\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)k + \bar{y}(t)$ is a solution of (1.1) for any constant vector k. Now to prove that every solution is of the form, let y(t) be any solution of (1.1) and $\bar{y}(t)$ be a particular solution of (1.1). Then $(y(t) - \bar{y}(t))$ is a solution of the homogeneous equation (2.1). Any solution of the homogeneous system (2.1) is of the form

$$y(t) - \bar{y}(t) = \left(P^T(t)\otimes Q^T(t)\right)\Phi(t)k$$

or $y(t) = \left(P^T(t)\otimes Q^T(t)\right)\Phi(t)k + \bar{y}(t)$.

Hence, any solution of the non-linear kronecker product system (1.1) is of the form

$$y(t) = \left(P^T(t)\otimes Q^T(t)\right)\Phi(t)k$$

$$+\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)\int_a^t \Phi^{-1}(s)$$

$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s, y(s))\ ds.$$

[III] FOUR-POINT BOUNDARY VALUE PROBLEM

In this section, we obtain our main result on existence and uniqueness of solutions associated with kronecker product four-point boundary value problem in terms of an integral equation involving Green's matrix.

Def.3.1: If $\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)k$ is a fundamental matrix of (1.1), then the matrix D defined by

$$D=\left(M_1\otimes N_1\right)\left(P^T(a)\otimes Q^T(a)\right)\Phi(a)+$$
$$\left(M_2\otimes N_2\right)\left(P^T(b)\otimes Q^T(b)\right)\Phi(b)+$$
$$\left(M_3\otimes N_3\right)\left(P^T(c)\otimes Q^T(c)\right)\Phi(c)+$$
$$\left(M_4\otimes N_4\right)\left(P^T(d)\otimes Q^T(d)\right)\Phi(d)$$

is called the characteristic matrix for the kronecker product boundary value problem (1.1) and (1.2).

Def 3.2: The dimension of the solution space of the kronecker product boundary value problem is the index of compatibility of the problem. A kronecker product boundary value problem is said to be incompatible if its index of compatibility is zero.

Theorem 3.1: Suppose the kronecker product homogeneous two-point boundary value problem is incompatible and there exists a constant K such that $\left\|f(t,y_1)-f(t,y_2)\right\|\le K\left\|(y_1-y_2)\right\|$ for all (t,y_1), $(t,y_2)\in[a, d]\times R^{n^2}$ and a constant $M > 0$ such that $\left\|G(t,s)\right\|\le M$ and further suppose that $MK(d-a) < 1$. Then there exists a unique solution of the kronecker product three point boundary value problem (1.1) & (1.2).

Proof : The general solution of (1.1) is of the form

$$y(t) = \left(P^T(t)\otimes Q^T(t)\right)\Phi(t)k$$

$$+\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)\int_a^t \Phi^{-1}(s)$$

$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s, y(s))\ ds.$$

Substituting the general form of y(t) in the boundary condition matrix (1.2), we get,

$$\left(M_1P^T(a)\otimes N_1Q^T(a)\right)\Phi(a)k+$$
$$\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)k+$$
$$\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)k+$$
$$\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)k$$

$$+\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)\int_a^b \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds$$
$$+\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)\int_a^c \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds$$

$$+\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)\int_a^d \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds = \alpha$$

$k = D^{-1}\alpha$ -

$$D^{-1}\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)\int_a^b \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds$$
-
$$D^{-1}\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)\int_a^c \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds$$
-
$$D^{-1}\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)\int_a^d \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds$$

Substituting the form of k in the general solution of y(t) in (1.1), we get

$$y(t) = \left(P^T(t)\otimes Q^T(t)\right)\Phi(t)\,D^{-1}\alpha -$$
$$\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)$$
$$[\,D^{-1}\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)\int_a^b \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds$$
-
$$D^{-1}\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)\int_a^c \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds$$
-
$$D^{-1}\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)\int_a^d \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds\,]$$
$$+\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)\int_a^t \Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1} f(s,y(s))ds.$$
$$= \int_a^d G(t,s)\, f(s,y(s))ds +$$
$$\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)\,D^{-1}\alpha,$$

where G(t,s) the Green's matrix, is given by

G(t,s)=

$$\begin{cases} \left(P^T(t)\otimes Q^T(t)\right)\Phi(t)D^{-1}\left(M_1P^T(a)\otimes N_1Q^T(a)\right)\Phi(a)\Phi^{-1}(s) \\ \left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}, a < s < t \le b < c < d \\ \\ -\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)D^{-1}[\;\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)+ \\ \left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)+\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)\;]\Phi^{-1}(s) \\ \left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}, a \le t < s < b < c < d \\ \\ -\left(P^T(t)\otimes Q^T(t)\right)\Phi(t)D^{-1}\begin{bmatrix}\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)+ \\ \left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)\end{bmatrix}\Phi^{-1}(s) \\ (P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a \le t < b < s < c < d \\ \\ \left(P^T(t)\otimes Q^T(t)\right)\Phi(t)D^{-1}\left[\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)\right]\Phi^{-1}(s) \\ (P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a \le t < b < c < s < d \end{cases}$$

where t$\in [a,b]$

G(t,s)=

$$\begin{cases} (P^T(t)\otimes Q^T(t))\Phi(t)\begin{bmatrix} I - D^{-1}(M_3P^T(c)\otimes N_3Q^T(c))\Phi(c) - \\ D^{-1}(M_4P^T(d)\otimes N_4Q^T(d))\Phi(d)\end{bmatrix} \\ \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a<b<s<t\le c<d \\ -(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}\begin{bmatrix}(M_3P^T(c)\otimes N_3Q^T(c))\Phi(c)+ \\ (M_4P^T(d)\otimes N_4Q^T(d))\Phi(d)\end{bmatrix} \\ \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a<b\le t<s<c<d \\ -(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}(M_4P^T(d)\otimes N_4Q^T(d))\Phi(d) \\ \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a<b<t\le c<s<d \\ (P^T(t)\otimes Q^T(t))\Phi(t)[I-D^{-1}(M_2P^T(b)\otimes N_2Q^T(b))\Phi(b)] \\ \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a<s<b<t\le c<d \end{cases}$$

where t$\in [b,c]$

G(t,s)=

$$\begin{cases} (P^T(t)\otimes Q^T(t))\Phi(t)\begin{bmatrix} I - D^{-1}(M_4P^T(d)\otimes N_4Q^T(d)) \\ \Phi(d)\end{bmatrix}\Phi^{-1}(s) \\ (P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a<b<c<s<t\le d \\ -(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}(M_4P^T(d)\otimes N_4Q^T(d))\Phi(d) \\ \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a<b<c\le t<s<d \\ (P^T(t)\otimes Q^T(t))\Phi(t)\begin{bmatrix} I - D^{-1}(M_3P^T(c)\otimes N_3Q^T(c))\Phi(c) \\ -D^{-1}(M_4P^T(d)\otimes N_4Q^T(d))\Phi(d)\end{bmatrix} \\ \Phi^{-1}(s)(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a<b<s<c<t\le d \\ -(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}(M_1P^T(a)\otimes N_1Q^T(a))\Phi(a)\Phi^{-1}(s) \\ (P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}, a<s<b<c<t\le d \end{cases}$$

where t$\in [c,d]$

Let S be a closed subset of a Banach space B. Define an operator H: S→S by

$$H(y^{(i)}(t)) = \int_a^d G(t,s)\, f(s, y^{(i-1)}(s))ds + (P^T(t)\otimes Q^T(t))\Phi(t)\, D^{-1}\alpha.$$ Then

$$\|H(y^{(i)}(t)) - H(y^{(i-1)}(t))\| \le \int_a^d \|G(t,s)\| \left[\|f(s,y^{(i-1)}(s)) - f(s,y^{(i-2)}(s))\|\right] ds.$$

$$\le \text{ MK } \|y^{(i-1)}(s) - y^{(i-2)}(s)\| \text{ (d-a)}$$

. . . .

$$\le M^{(i-1)}K^{(i-1)}(d-a)^{(i-1)}\|y^{(1)}(s) - y^{(0)}(s)\|$$

where M, K are positive constants.

Thus if $MK(d-a) < 1$, H is a contraction operator. Hence by the Banach fixed point theorem, H has a unique fixed point and this fixed point is the unique solution of the three point kronecker product boundary value problem (1.1) and (1.2).

Theorem 3.2 : The Green's matrix G(t,s) has the following properties :

(i) The components of G(t,s) regarded as functions of t with s fixed have continuous first derivatives everywhere except at t = s. At the point t = s, G has an upward jump-discontinuity of magnitude $(P^T(t)\otimes Q^T(t))\,(P(t)P^T(t)\otimes Q(t)Q^T(t))^{-1}$.

i.e., $G(s^+,s) - G(s^-,s) = (P^T(s)\otimes Q^T(s))\,(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}$.

(ii) G(t,s) is a formal solution of the kronecker product homogeneous boundary value problem (2.1) satisfying (1.2). G fails to be a true solution because of its discontinuity at t = s.

(iii) G(t,s) satisfying properties (i) and (ii) is unique.

Proof : First we prove for t $\in [a,b]$. For fixed s, the components of G (t, s) have continuous first derivatives with respect to t on each of the subintervals [a,s) and (s,b]. Now consider

$G(s^+,s)$-$G(s^-,s)$=

$$(P^T(s)\otimes Q^T(s))\Phi(s)D^{-1}\,(M_1P^T(a)\otimes N_1Q^T(a))\Phi(a)\,\Phi^{-1}(s)\,(P(s)P^T(s)\otimes Q(s)Q^T(s))^{-1}$$

$$+(P^T(s)\otimes Q^T(s))\Phi(s)D^{-1}$$

$$(M_2P^T(b)\otimes N_2Q^T(b)\Phi(b)) + (M_3P^T(c)\otimes N_3Q^T(c)\Phi(c)) + (M_4P^T(d)\otimes N_4Q^T(d)\Phi(d))]$$

$=\left(P^T(s)\otimes Q^T(s)\right)\Phi(s)\,\Phi^{-1}(s)$

$D^{-1}D\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}$

$= \left(P^T(s)\otimes Q^T(s)\right)\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}$

ii) The representation of G(t,s) clearly shows that G(t,s) is a matrix solution of kronecker product homogeneous system (2.1) as [a,s) and (s,b]. We show that G(t,s) satisfies the given boundary condition matrix (1.2), we have

$\left(M_1\otimes N_1\right)$ G(a,s)+ $\left(M_2\otimes N_2\right)$ G(b,s)+ $\left(M_3\otimes N_3\right)$ G (c,s) + $\left(M_4\otimes N_4\right)$ G(d,s)

$$=D-\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)$$
$$+\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)$$
$$+\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)\;)\,H_-$$
$$+(\;\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)$$
$$+\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)$$
$$+(M_4P^T(d)\otimes N_4Q^T(d))\Phi(d)\;)\;H_+$$
$$=DH_- -(\;\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)$$
$$+\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)+$$
$$(M_4P^T(d)\otimes N_4Q^T(d))\Phi(d))\;)\,H_-$$
$$+(\;\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)$$
$$+\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)$$
$$+\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)\;)\,H_+$$
$$=-DD^{-1}[\;\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)$$
$$+\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)$$
$$+\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)]\,\Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}$$
$$+[\;\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)$$
$$+\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)$$
$$+\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)]\,(H_+ - H_-) = 0.$$

where, $H^+ = D^{-1}\left(M_1P^T(a)\otimes N_1Q^T(a)\right)\Phi(a)$

$$\Phi^{-1}(s)\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}$$

$$H^- = -D^{-1}[\;\left(M_2P^T(b)\otimes N_2Q^T(b)\right)\Phi(b)$$
$$+\left(M_3P^T(c)\otimes N_3Q^T(c)\right)\Phi(c)$$
$$+\left(M_4P^T(d)\otimes N_4Q^T(d)\right)\Phi(d)]\,\Phi^{-1}(s)$$
$$\left(P(s)P^T(s)\otimes Q(s)Q^T(s)\right)^{-1}$$

Similarly, when $t\in[b,c]$ and $t\in[c,d]$

$\left(M_1\otimes N_1\right)$ G(a,s)+ $\left(M_2\otimes N_2\right)$ G(b,s)+ $\left(M_3\otimes N_3\right)$ G(c,s) + $\left(M_4\otimes N_4\right)$ G(d,s) = 0.

Thus G is a formal solution of the homogeneous kronecker product boundary value problem.

iii) Now to prove that G is unique, let G_1(t,s) and G_2(t,s) be continuous matrices with properties (i), (ii). Write H(t,s) = G_1(t,s) - G_2(t,s). Clearly G is continuous on [a,s) and (s,b], and H satisfies the kronecker product homogeneous system (2.1) on [a,s) and (s,b]. At the point t = s.

$H\left(s^+,s\right) - H\left(s^-,s\right) =$

$G_1\left(s^+,s\right) - G_2\left(s^+,s\right) - G_1\left(s^-,s\right) + G_2\left(s^-,s\right) =$

$[G_1\left(s^+,s\right) - G_1\left(s^-,s\right)] - [G_2\left(s^+,s\right) - G_2\left(s^-,s\right)] = 0$

Therefore H has a removable discontinuity at t = s. By defining H appropriately at this point, we ensure that it is continuous for all t $\in$[a,b]. Since the boundary condition matrix is linear and H is a linear combination of G_1 and G_2, we have

$$\left(M_1\otimes N_1\right)H(a,s)+\left(M_2\otimes N_2\right)H(b,s)+$$
$$\left(M_3\otimes N_3\right)H(c,s)+\left(M_4\otimes N_4\right)H(d,s)=0$$

Similarly, when $t\in[b,c]$ and $t\in[c,d]$

$\left(M_1\otimes N_1\right)H(a,s)+\left(M_2\otimes N_2\right)H(b,s)+$ $\left(M_3\otimes N_3\right)H(c,s)+\left(M_4\otimes N_4\right)H(d,s)=$ 0.Since H is a solution of (2.1), it satisfies the homogeneous boundary condition matrix and from our initial assumption that the homogeneous three point boundary value problem has only a trivial solution, it follows that H (t,s) = 0.

i.e.,$G_1(t,s) - G_2(t,s) = 0$ implies $G_1(t,s) = G_2(t,s)$.

Thus G is unique.

[IV] N - POINT BOUNDARY VALUE PROBLEM

In this section, we consider the Multi-Point Boundary Value Problem

$$Ly = (P(t)\otimes Q(t))y'(t) + (R(t)\otimes S(t))y(t) = f(t,y(t)), \quad (4.1)$$

with the boundary condition

$$\sum_{i=1}^{n}(M_i \otimes N_i)y(t_i) = \alpha, \quad (4.2)$$

where, $a = t_1 < t_2 < .... < t_n = d$, where the matrices $P,Q,R,S,M_1, M_2,M_3,M_4,N_1,N_2,N_3,N_4$ are same as in section 1.

Theorem 4.1 : Suppose the kronecker product multi-point boundary value problem is incompatible and there exists a constant K > 0 such that $\|f(t,y_1) - f(t,y_2)\| \le K\|(y_1 - y_2)\|$

for all (t,y_1), $(t,y_2) \in [a, d] \times R^{n^2}$ and a constant M > 0 such that $\|G(t,s)\| \le M$ and further suppose that MK (d-a) < 1. Where

G(t,s)=

$$\Big\{(P^T(t)\otimes Q^T(t))\Phi(t)D^{-1}\sum_{i=1}^{l}\big(M_iP^T(\xi_i)\otimes N_iQ^T(\xi_i)\big)$$

$$\Phi(\xi_i)\Phi^{-1}(s)^{-1}\big(P(s)P^T(s)\otimes Q(s)Q^T(s)\big)^{-1}, t \le s$$

$$-\big(P^T(t)\otimes Q^T(t)\big)\Phi(t)D^{-1}\sum_{i=l+1}^{n}\big(M_iP^T(\xi_i)\otimes N_iQ^T(\xi_i)\big)$$

$$\Phi(\xi_i)\Phi^{-1}(s)\big(P(s)P^T(s)\otimes Q(s)Q^T(s)\big)^{-1}, t \ge s$$

$$for, t \in \lfloor \xi_l, \xi_{l+1} \rfloor, (1 \le l \le n-1),$$

then there existes a unique solution of the the kronecker product multi-point boundary value problem (4.1),(4.2) and the unique solution is given by

$$\int_a^d G(t,s)\, f(s,y(s))ds + \big(P^T(t)\otimes Q^T(t)\big)\Phi(t)\, D^{-1}\alpha$$

Suppose the characteristic matrix

$$D = \sum_{i=1}^{n}\big(M_iP^T(\xi_i)\otimes N_iQ^T(\xi_i)\big)\Phi(\xi_i)$$

is a rectangular matrix. Now the solution of (4.1) is of the form

$$y(t) = \big(P^T(t)\otimes Q^T(t)\big)\Phi(t)\,k +$$

$$+\big(P^T(t)\otimes Q^T(t)\big)\Phi(t)\int_a^t \Phi^{-1}(s)$$

$$\big(P(s)P^T(s)\otimes Q(s)Q^T(s)\big)^{-1} f(s, y(s))\, ds.$$

Substituting the general form of y(t) in the boundary condition matrix (4.2), we get,

$k = D^{-1}\alpha$ -

$$D^{-1}\sum_{1=2}^{n}\big(M_iP^T(\xi_i)\otimes N_iQ^T(\xi_i)\big)\Phi(\xi_1)$$

$$\int_a^{t_i}\Phi^{-1}(s)\big(P(s)P^T(s)\otimes Q(s)Q^T(s)\big)^{-1} f(s,y(s))ds.$$

Substituting the form of k in the general solution of y(t) in (4.1), we get

$$y(t) = \big(P^T(t)\otimes Q^T(t)\big)\Phi(t)\, D^{-1}[\alpha -$$

$$\sum_{1=2}^{n}\big(M_iP^T(\xi_i)\otimes N_iQ^T(\xi_i)\big)\Phi(\xi_1)$$

$$\int_a^{t_i}\Phi^{-1}(s)\big(P(s)P^T(s)\otimes Q(s)Q^T(s)\big)^{-1} f(s,y(s))ds]$$

$$+\big(P^T(t)\otimes Q^T(t)\big)\Phi(t)\int_a^t \Phi^{-1}(s)$$

$$\big(P(s)P^T(s)\otimes Q(s)Q^T(s)\big)^{-1} f(s, y(s))\, ds.$$

Example 4.1: Consider the four point boundary value problem

$$(P(t)\otimes Q(t))y'(t) + (R(t)\otimes S(t))y(t) = f(t,y(t)) \quad (4.3)$$

Where, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

$R = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $S = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

$$A = \left(PP^T \otimes QQ^T\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \left(PP^{T'} \otimes QQ^{T'}\right) + \left(RP^T \otimes SQ^T\right)$$

$$= \begin{bmatrix} 0 & -3 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Now the transformation $y(t) = \left(P^T \otimes Q^T\right) z(t)$.

The equation (4.3) becomes

$$Az' + Bz = 0 \qquad (4.4)$$

i.e., $$z' = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} z.$$

The fundamental matrix $\Phi(t)$ for the system (4.4) is given by

$$\Phi(t) = \begin{bmatrix} 1 & 0 & 0 & e^{3t} \\ 0 & 0 & 0 & e^{3t} \\ 0 & t & e^t & 0 \\ 0 & 0 & e^t & 0 \end{bmatrix} \qquad f(t,\ y(t)) = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \\ 0 \end{bmatrix}$$

The characteristic matrix D is given by

$$D = (M_1 \otimes N_1)\left(P^T \otimes Q^T\right)\Phi(0) + (M_2 \otimes N_2)\left(P^T \otimes Q^T\right)\Phi(1) + (M_3 \otimes N_3)\left(P^T \otimes Q^T\right)\Phi(2) + (M_4 \otimes N_4)\left(P^T \otimes Q^T\right)\Phi(3)$$

where, $M_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $N_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

$M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $N_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

$M_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $N_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

$M_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $N_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

$$D = \begin{bmatrix} 1 & 1 & 2.72 & 423.5 \\ 2 & 1 & 2.72 & 423.5 \\ 0 & 1 & 3.72 & 0 \\ 0 & 1 & 4.72 & 0 \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & P & Q \end{bmatrix}$$

where A = -1, B = 1, C = 0, D = 0, E = 0, F = 0, G = 4.72, H = -3.72, I = 0, J = 0, K = -1, L = 1, M = 0.00472, N = 0.00361, P = -0.00472, Q = 0.00361.

Now the solution will be in the form

$$y(t) = \int_0^3 G(t,s) f(s, y(s)) ds = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \end{bmatrix},$$

$$A_1 = (3M + 8N)e^{4t} + (P + Q)s^{-1}e^{2t} - 3(M + N)s^{-1}e^{2t}$$

$$A_2 = 2(A + B - 2)e^t + 4(B + N)e^{4t} - 2Ae^{-t} - 2\left(M + Ns^{-1}\right)e^{2t} - (A + 3B)s^{-1}e^{-t} - (P + Q)s^{-1}e^{2t} - (M + N)e^{2t},$$

$A_3 = 0$, $A_4 = -2(K + L)s^{-1} - (G + H)s^{-1}$,

$$A_5 = (M + N)e^{4t} - 2(Gt + H)s^{-1}e^{-t} - 2(K + L)s^{-1} + 4ts^{-1}e^{-t} - (G + H)ts^{-1}e^{-t} - (K + L)s^{-1} + (M + N)s^{-1}e^{2t} + (P + Q)s^{-1}e^{2t}$$

$A_6 = 0$, $A_7 = 0$, $A_8 = 0$, $A_9 = 0$.

and $$G(t,s) = \begin{bmatrix} P_1 & Q_1 & R_1 & S_1 \\ P_2 & Q_2 & R_2 & S_2 \\ P_3 & Q_3 & R_3 & S_3 \\ P_4 & Q_4 & R_4 & S_4 \\ P_5 & Q_5 & R_5 & S_5 \\ P_6 & Q_6 & R_6 & S_6 \\ P_7 & Q_7 & R_7 & S_7 \\ P_8 & Q_8 & R_8 & S_8 \\ P_9 & Q_9 & R_9 & S_9 \end{bmatrix}$$

where, $P_1 = (3M + 8N)e^{3t}$,

$$Q_1 = -\begin{pmatrix} (3M+N)e^{3t} + (3M+N)e^{3(1-s+t)} - 4e^{3(t-s)} \\ +2e^{(6-3s)} + 5Ne^{3t} - 2Ne^{3(2t-s)} \end{pmatrix}$$

$$+\left((M+N)e^{3(2-s+t)}\right)$$

$$R_1 = (P+Q)s^{-1}e^{3t} - 3(M+N)s^{-1}e^{3t},$$

$$S_1 = 2Qe^{(3t-s)} + (M+N)s^{-1}e^{3t} - (P+Q)s^{-1}e^{3t} + (P+Q)e^{(3t+1-s)}$$
$$-(M+N)e^{(3t+1-s)}$$

$$P_2 = 2(A+B-2) + 4(B+N)e^{3t}$$

$$Q_2 = \begin{pmatrix} (4-3A-3B) - 2(A+B)e^{3(1-s)} - 3(A+B)e^{3(2-s)} \\ -(3M+4N)e^{3t} - (A+B)e^{(3-s)} \end{pmatrix}$$

$$+\left(4e^{3(2-s)} - 2(M+N)e^{3(1+t-s)} - (M+N)e^{3(t+2-s)}\right),$$

$$R_2 = -2A + \begin{pmatrix} -2(M+Ns^{-1})e^{3t} - (A+3B)s^{-1} \\ -(M+N)s^{-1}e^{3t} - (P+Q)s^{-1}e^{3t} \end{pmatrix},$$

$$S_2 = 2(A+B)s^{-1} - (A+NB) - 2(A+B)e^{(1-S)} +$$
$$3(M+N)s^{-1}e^{3t} - 3(M+N)e^{(3t+1-s)}$$

$$+(P+Q)s^{-1}e^{3t} + (P+Q)e^{(3t+1-S)},$$

$P_3 = Q_3 = R_3 = S_3 = 0$, $P_4 = Q_4 = 0$,

$$R_4 = -2(K+L)s^{-1}e^{t} + (G+H)e^{t}s^{-1},$$

$$S_4 = (K+L)e^{(t-s)} + 2(K+L)s^{-1}e^{t} + (G+H)e^{(t-s)}$$
$$-2(K+L)e^{(1+t-s)} + 2e^{(s-1)}$$
$$-(G+H)e^{(t+1-S)},\quad P_5 = (M+N)e^{3t}$$

$$Q_5 = 4e^{3(t-s)} - (M+N)e^{3t} - (M+N)e^{3(1+t-s)},$$

$$R_5 = 4ts^{-1} - 2(Gt+H)s - 2(K+L)s^{-1}e^{t} - (G+H)ts^{-1}$$
$$+(K+L)s^{-1}e^{t} + (M+N)s^{-1}e^{3t}$$
$$+(P+Q)s^{-1}e^{3t},$$

$$S_5 = (G+H)ts^{-1} + (Lt+H)e^{-S} + 2Hs^{-1} + (3G+H)ts^{-1}$$
$$+3(K+L)s^{-1}e^{t} - 2He^{(1-s)}$$
$$-(M+N+P+Q)s^{-1}e^{3t} - (3G+H)ts^{(1-s)} - 3(K+L)e^{(1+t-s)}$$
$$+(M+N+P+Q)e^{(1-s)}$$

$P_6 = Q_6 = R_6 = S_6 = 0$,

$P_7 = Q_7 = R_7 = S_7 = 0$ $P_8 = Q_8 = R_8 = S_8 = 0$,

$P_9 = Q_9 = R_9 = S_9 = 0$

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