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A STUDY ON GLOBAL STABILITY OF A FOUR SPECIES SYN ECO-SYSTEM WITH COMMENSAL PREY-PREDATOR PAIR WITH PREY-PREDATOR PAIR OF HOSTS

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ABSTRACT

The System comprises of a Prey (S_1), a Predator (S_2) that survives upon S_1, two Hosts S_3 and S_4 for which S_1, S_2 are Commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 is Prey for S_4 and S_4 is Predator for S_3. Limited resources are considered for all the four species in this case The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In this paper we establish the global stability of a Four Species Syn Eco-System with Commensal Prey-Predator pair with Prey-Predator pair of Hosts by constructing a suitable Liapunov's function. Further the numerical solutions for the growth rate equations are computed using Runga-Kutta fourth order method. Some observations are identified from the relationship between the initial value and the carrying capacity of four species.

Keywords: Equilibrium state, Globally asymptotically stable, Predator, Prey, Liapunov's function, Commensalism.

AMS Classification: 92D25, 92D40

INTRODUCTION

Ecology is the study of living beings such as plants and animals in relation to their habitats and habits. It mainly deals with the evolutionary biology which explains us about how the living being is regulated in nature. It is natural that two or more species living in a common habitat interact in different ways. Significant researches in the area of theoretical ecology has been initiated by Lotka [21] in 1925 and by Volterra [27] in 1931. Since then, several mathematicians and ecologists contributed to the growth of this area of knowledge. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and Parasitism.

Mathematical modeling has been playing an important role for the last half a century in explaining several phenomena concerned with individuals and groups of populations in nature. The general concept of modeling has been presented in the treatises of Meyer[22], Kushing[16], Paul[23], Kapur[17,18].

Srinivas[26] studied competitive ecosystem of two species and three species with limited and unlimited resources. Later, Lakshminarayan [19], Laxminarayan and Pattabhi Ramacharyulu [20] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Krishna Gandhi [5] and by Bhaskara Rama Sarma and Pattabhi Ramacharyulu[6], while Ravindra Reddy[25] investigated mutualism between two species. Acharyulu K.V.L.N and Pattabhi Ramacharyulu [1-4] derived some productive results on various mathematical models of ecological Ammensalism with multifarious resources in the manifold directions. Further Phani Kumar [24] studied some mathematical models of ecological commensalism. The present authors Hari Prasad and Pattabhi Ramacharyulu [7-15] discussed on the stability of a four species syn-ecosystem.

Predation: Predation is a relationship between two species where one species kills and devours other for food. The species which kills other is called a predator and the species which is killed is called a prey. A common example for predation is a cat killing a rat.

Commensalism: Commensalism is a symbiotic interaction between two or more populations which live together, and which only one of the populations is benefited while the other is not effected. Remora living with a Shark is an example for the Commensalism.

Some real-life examples of a Syn-Eco-System with Commensal Prey-Predator pair with Pray-Predator pair of Hosts are given in the following Table.1.

Table.1

Sl. No.	Examples of S_1	Examples of S_2	Examples of S_3	Examples of S_4
1	Infusoria	Sea anemone	Arthopods	Clown fish
2	Small beetle	Remora	Fish (or) small aquatic vertebrate	Shark
3	Rabit	Golden Jackal	Deer	Tiger
4	Insects	Army Ants	Earth worms	Birds
5	Grass	Cow	Insects	Cattle egrets

A Schematic Sketch of the system under investigation is shown here under Fig. 1.

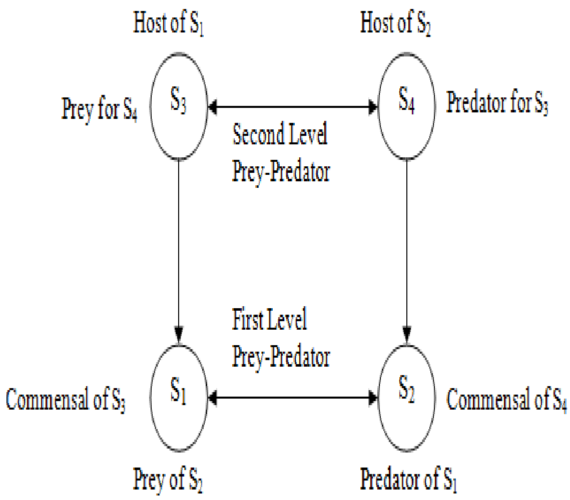

Fig. 1 Schematic Sketch of the Syn Eco - System

BASIC EQUATIONS

The model equations for a four species syn eco-system is given by the following system of first order non-linear ordinary differential equations employing the following notation.

S_1 : Prey for S_2 and commensal for S_3.

S_2 : Predator surviving upon S_1 and commensal for S_4.

S_3 : Host for the commensal (S_1) and Prey for S_4.

S_4 : Host of the commensal (S_2) and Predator surviving upon S_4.

$N_i(t)$: The Population strength of S_i at time t, i = 1, 2, 3, 4

t : Time instant

a_i : Natural growth rate of S_i, i = 1, 2, 3, 4

a_{ii} : Self inhibition coefficient of S_i, i = 1, 2, 3, 4

a_{12}, a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1

a_{34}, a_{43} : Interaction (Prey-Predator) coefficients of S_3 due to S_4 and S_4 due to S_3

a_{13}, a_{24} : Coefficients of commensal for S_1 due to the Host S_3 and S_2 due to the Host S_4

$K_i = \frac{a_i}{a_{ii}}$: Carrying capacities of S_i, i = 1, 2, 3, 4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24}, a_{34}, a_{43} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 + a_{13} N_1 N_3 \quad (1)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{21} N_1 N_2 + a_{24} N_2 N_4 \quad (2)$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 - a_{34} N_3 N_4 \quad (3)$$

$$\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^2 + a_{43} N_3 N_4 \quad (4)$$

EQUILIBRIUM STATES

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0,\ i = 1, 2, 3, 4 \quad (5)$$

(i). Fully washed out state

$$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = 0$$

(ii). Semi/partially washed out states

States in which three of the four species are washed out and fourth is not.

$$E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = K_4$$

$$E_3 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = K_3, \bar{N}_4 = 0$$

$$E_4 : \bar{N}_1 = 0, \bar{N}_2 = K_2, \bar{N}_3 = 0, \bar{N}_4 = 0$$

$$E_5 : \bar{N}_1 = K_1, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = 0$$

Only two of the four species are washed out while the other two are not.

$$E_6 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = \frac{\alpha}{\beta}, \bar{N}_4 = \frac{\gamma}{\beta}$$

where

$$\alpha = a_3 a_{44} - a_4 a_{34},\ \beta = a_{33} a_{44} + a_{34} a_{43} > 0, \gamma = a_3 a_{43} + a_4 a_{33} > 0$$

This would exist only when $a_3 a_{44} > a_4 a_{34}$

$$E_7 : \bar{N}_1 = 0, \bar{N}_2 = \frac{\delta_1}{a_{22} a_{44}}, \bar{N}_3 = 0, \bar{N}_4 = K_4$$

where $\delta_1 = a_2 a_{44} + a_4 a_{24} > 0$

$$E_8 : \bar{N}_1 = 0, \bar{N}_2 = K_2, \bar{N}_3 = K_3, \bar{N}_4 = 0$$

$$E_9 : \bar{N}_1 = K_1, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = K_4$$

$$E_{10} : \bar{N}_1 = \frac{\delta_2}{a_{11} a_{33}}, \bar{N}_2 = 0, \bar{N}_3 = K_3, \bar{N}_4 = 0$$

where $\delta_2 = a_1 a_{33} + a_3 a_{13} > 0$

$$E_{11} : \bar{N}_1 = \frac{\alpha_1}{\beta_1}, \bar{N}_2 = \frac{\gamma_1}{\beta_1}, \bar{N}_3 = 0, \bar{N}_4 = 0$$

where $\alpha_1 = a_1 a_{22} - a_2 a_{12}$,

$\beta_1 = a_{11} a_{22} + a_{12} a_{21} > 0$,

$\gamma_1 = a_1 a_{21} + a_2 a_{11} > 0$

This would exist only when $a_1 a_{22} > a_2 a_{12}$

Only one of the four species is washed out while the other three are not.

$$E_{12} : \bar{N}_1 = 0, \bar{N}_2 = \frac{a_2 \beta + a_{24} \gamma}{a_{22} \beta}, \bar{N}_3 = \frac{\alpha}{\beta}, \bar{N}_4 = \frac{\gamma}{\beta}$$

$$E_{13} : \bar{N}_1 = \frac{a_1 \beta + a_{13} \alpha}{a_{11} \beta}, \bar{N}_2 = 0, \bar{N}_3 = \frac{\alpha}{\beta}, \bar{N}_4 = \frac{\gamma}{\beta}$$

$$E_{14} : \bar{N}_1 = \frac{a_1 a_{22} a_{44} - a_{12}\delta_1}{a_{44}\beta_1}, \bar{N}_2 = \frac{a_1 a_{21} a_{44} + a_{11}\delta_1}{a_{44}\beta_1}, \bar{N}_3 = 0, \bar{N}_4 = K_4$$

This would exist only when $a_1 a_{22} a_{44} - a_{12}\delta_1 > 0$

$$E_{15} : \bar{N}_1 = \frac{a_{22}\delta_2 - a_2 a_{12} a_{33}}{a_{33}\beta_1}, \bar{N}_2 = \frac{a_{21}\delta_2 + a_2 a_{11} a_{33}}{a_{33}\beta_1}, \bar{N}_3 = K_3, \bar{N}_4 = 0$$

This would exist only when $a_{22}\delta_2 - a_2 a_{12} a_{33} > 0$

(iii). The normal steady state.

$$E_{16} : \bar{N}_1 = \frac{a_{22}\alpha_2 - a_{12}\gamma_2}{\beta_1}, \bar{N}_2 = \frac{a_{11}\gamma_2 + a_{21}\alpha_2}{\beta_1}, \bar{N}_3 = \frac{\alpha}{\beta}, \bar{N}_4 = \frac{\gamma}{\beta}$$

where $\alpha_2 = a_1 + a_{13}\frac{\alpha}{\beta}$, $\gamma_2 = a_2 + a_{24}\frac{\gamma}{\beta} > 0$

This would exist only when $a_{22}\alpha_2 - a_{12}\gamma_2 > 0$

LIAPUNOV'S FUNCTION FOR GLOBAL STABILITY

The present authors discussed earlier [11-15] the local stability of all above sixteen equilibrium states but only $E_7\left(0, \bar{N}_2, 0, \bar{N}_4\right)$, $E_{12}\left(0, \bar{N}_2, \bar{N}_3, \bar{N}_4\right)$, $E_{13}\left(\bar{N}_1, 0, \bar{N}_3, \bar{N}_4\right)$, $E_{14}\left(\bar{N}_1, \bar{N}_2, 0, \bar{N}_4\right)$ and $E_{16}\left(\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4\right)$ were found to be stable.. In the present paper, the global stability of dynamical system (1), (2), (3) and (4) at these five equilibrium states are examined by suitable Liapunov's functions.

Theorem 1. *The equilibrium point* $E_7\left(0, \bar{N}_2, 0, \bar{N}_4\right)$ *is globally asymptotically stable.*

Proof : Let us consider the following Liapunov's function

$$V(N_2, N_4) = N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right) + d_1\left[N_4 - \bar{N}_4 \ln\left(\frac{N_4}{\bar{N}_4}\right)\right] \quad (6)$$

where d_1 is a suitable constant to be determined as in the subsequent steps.

Now, the time derivative of V, along with solution of (2) and (4) can be written as

$$\frac{dV}{dt} = \left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + d_1\left(\frac{N_4 - \bar{N}_4}{N_4}\right)\frac{dN_4}{dt} \quad (7)$$

$$= \left(N_2 - \bar{N}_2\right)\left(a_2 - a_{22}N_2 + a_{24}N_4\right) + d_1\left(N_4 - \bar{N}_4\right)\left(a_4 - a_{44}N_4\right)$$

$$= \left(N_2 - \bar{N}_2\right)\left(a_{22}\bar{N}_2 - a_{24}\bar{N}_4 - a_{22}N_2 + a_{24}N_4\right) + d_1\left(N_4 - \bar{N}_4\right)\left(a_{44}\bar{N}_4 - a_{44}N_4\right)$$

$$= -a_{22}\left(N_2 - \bar{N}_2\right)^2 + a_{24}\left(N_2 - \bar{N}_2\right)\left(N_4 - \bar{N}_4\right) + d_1\left[-a_{44}\left(N_4 - \bar{N}_4\right)^2\right]$$

$$\left.\begin{aligned} \frac{dV}{dt} &= -\left[\sqrt{a_{22}}\left(N_2 - \bar{N}_2\right) - \sqrt{d_1 a_{44}}\left(N_4 - \bar{N}_4\right)\right]^2 \\ &\quad - \left[\left(2\sqrt{d_1 a_{22} a_{44}} - a_{24}\right)\left(N_2 - \bar{N}_2\right)\left(N_4 - \bar{N}_4\right)\right] \end{aligned}\right\} \quad (8)$$

The constant d_1 as so chosen that, the coefficient of $\left(N_2 - \bar{N}_2\right)\left(N_4 - \bar{N}_4\right)$ in (8) vanish.

Then we have $d_1 = \frac{a_{24}^2}{4a_{22}a_{44}} > 0$, with this choice of a constant d_1

$$\frac{dV}{dt} = -\left[\sqrt{a_{22}}\left(N_2 - \bar{N}_2\right)^2 - \frac{a_{24}}{2\sqrt{a_{22}}}\left(N_4 - \bar{N}_4\right)\right]^2 \quad (9)$$

which is negative definite.

Hence, $E_7\left(0, \bar{N}_2, 0, \bar{N}_4\right)$ is globally asymptotically stable.

Theorem 2. *The equilibrium point $E_{12}\left(0,\bar{N}_2,\bar{N}_3,\bar{N}_4\right)$ is globally asymptotically stable.*

Proof : Let us consider the following Liapunov's function

$$\left.\begin{aligned} V(N_2,N_3,N_4) &= N_2-\bar{N}_2-\bar{N}_2\ln\left(\frac{N_2}{\bar{N}_2}\right)+d_1\left[N_3-\bar{N}_3-\bar{N}_3\ln\left(\frac{N_3}{\bar{N}_3}\right)\right] \\ &\quad +d_2\left[N_4-\bar{N}_4-\bar{N}_4\ln\left(\frac{N_4}{\bar{N}_4}\right)\right] \end{aligned}\right\} \tag{10}$$

where d_1 and d_2 are suitable constants to be determined as in the subsequent steps.

Now, the time derivative of V, along with solutions of (2), (3) and (4) can be written as

$$\frac{dV}{dt}=\left(\frac{N_2-\bar{N}_2}{N_2}\right)\frac{dN_2}{dt}+d_1\left(\frac{N_3-\bar{N}_3}{N_3}\right)\frac{dN_3}{dt}+d_2\left(\frac{N_4-\bar{N}_4}{N_4}\right)\frac{dN_4}{dt} \tag{11}$$

$$=\left(N_2-\bar{N}_2\right)\left(a_2-a_{22}N_2+a_{24}N_4\right)+d_1\left(N_3-\bar{N}_3\right)\left(a_3-a_{33}N_3-a_{34}N_4\right)$$
$$+d_2\left(N_4-\bar{N}_4\right)\left(a_4-a_{44}N_4+a_{43}N_3\right)$$

$$=\left(N_2-\bar{N}_2\right)\left(a_{22}\bar{N}_2-a_{24}\bar{N}_4-a_{22}N_2+a_{24}N_4\right)$$

$$+d_1\left(N_3-\bar{N}_3\right)\left(a_{33}\bar{N}_3+a_{34}\bar{N}_4-a_{33}N_3-a_{34}N_4\right)$$

$$+d_2\left(N_4-\bar{N}_4\right)\left(a_{44}\bar{N}_4-a_{43}\bar{N}_3-a_{44}N_4+a_{43}N_3\right)$$

$$=-a_{22}\left(\bar{N}_2-\bar{N}_2\right)^2+a_{24}\left(N_2-\bar{N}_2\right)\left(N_4-\bar{N}_4\right)$$

$$+d_1\left[-a_{33}\left(N_3-\bar{N}_3\right)^2-a_{34}\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right)\right]$$

$$+d_2\left[-a_{44}\left(N_4-\bar{N}_4\right)^2+a_{43}\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right)\right]$$

$$\left.\begin{aligned} \frac{dV}{dt} &= d_1\left[-a_{33}\left(N_3-\bar{N}_3\right)^2+\left(\frac{d_2a_{43}}{d_1}-a_{34}\right)\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right)\right] \\ &\quad -\left[\sqrt{a_{22}}\left(N_2-\bar{N}_2\right)-\sqrt{d_2a_{44}}\left(N_4-\bar{N}_4\right)\right]^2-\left(2\sqrt{d_2a_{22}a_{44}}-a_{24}\right)\left(N_2-\bar{N}_2\right)\left(N_4-\bar{N}_4\right) \end{aligned}\right\} \tag{12}$$

The constants d_1 and d_2 as so chosen that, the coefficients of $\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right),\left(N_2-\bar{N}_2\right)\left(N_4-\bar{N}_4\right)$ in (12) vanish.

Then we have $d_1=\dfrac{a_{24}^2a_{43}}{4a_{22}a_{34}a_{44}}>0,\ d_2=\dfrac{a_{24}^2}{4a_{22}a_{44}}>0$, with this choice of the constants d_1 and d_2.

$$\frac{dV}{dt}=-\left[\frac{a_{24}^2a_{33}a_{43}}{4a_{22}a_{34}a_{44}}\left(N_3-\bar{N}_3\right)^2+\left\{\sqrt{a_{22}}\left(N_2-\bar{N}_2\right)-\frac{a_{24}}{2\sqrt{a_{22}}}\left(N_4-\bar{N}_4\right)\right\}^2\right] \tag{13}$$

which is negative definite.

Hence, $E_{12}\left(0,\bar{N}_2,\bar{N}_3,\bar{N}_4\right)$ is globally asymptotically stable.

Theorem 3. *The equilibrium point $E_{13}\left(\bar{N}_1,0,\bar{N}_3,\bar{N}_4\right)$ is globally asymptotically stable.*

Proof : Let us consider the following Liapunov's function

$$\left.\begin{aligned} V(N_1,N_3,N_4) &= N_1-\bar{N}_1-\bar{N}_1\ln\left(\frac{N_1}{\bar{N}_1}\right)+d_1\left[N_3-\bar{N}_3-\bar{N}_3\ln\left(\frac{N_3}{\bar{N}_3}\right)\right] \\ &\quad +d_2\left[N_4-\bar{N}_4-\bar{N}_4\ln\left(\frac{N_4}{\bar{N}_4}\right)\right] \end{aligned}\right\} \tag{14}$$

where d_1 and d_2 are suitable constants to be determined as in the subsequent steps.

Now, the time derivative of V, along with solutions of (1), (3) and (4) can be written as

$$\frac{dV}{dt}=\left(\frac{N_1-\bar{N}_1}{N_1}\right)\frac{dN_1}{dt}+d_1\left(\frac{N_3-\bar{N}_3}{N_3}\right)\frac{dN_3}{dt}+d_2\left(\frac{N_4-\bar{N}_4}{N_4}\right)\frac{dN_4}{dt} \quad (15)$$

$$=\left(N_1-\bar{N}_1\right)\left(a_1-a_{11}N_1+a_{13}N_3\right)+d_1\left(N_3-\bar{N}_3\right)\left(a_3-a_{33}N_3-a_{34}N_4\right)$$
$$+d_2\left(N_4-\bar{N}_4\right)\left(a_4-a_{44}N_4+a_{43}N_3\right)$$

$$=\left(N_1-\bar{N}_1\right)\left(a_{11}\bar{N}_1-a_{13}\bar{N}_3-a_{11}N_1+a_{13}N_3\right)$$

$$+d_1\left(N_3-\bar{N}_3\right)\left(a_{33}\bar{N}_3+a_{34}\bar{N}_4-a_{33}N_3-a_{34}N_4\right)$$

$$+d_2\left(N_4-\bar{N}_4\right)\left(a_{44}\bar{N}_4-a_{43}\bar{N}_3-a_{44}N_4+a_{43}N_3\right)$$

$$=-a_{11}\left(\bar{N}_1-\bar{N}_1\right)^2+a_{13}\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)$$

$$+d_1\left[-a_{33}\left(N_3-\bar{N}_3\right)^2-a_{34}\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right)\right]$$

$$+d_2\left[-a_{44}\left(N_4-\bar{N}_4\right)^2+a_{43}\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right)\right]$$

$$\frac{dV}{dt}=-\left\{\left[\left\{\sqrt{a_{11}}\left(N_1-\bar{N}_1\right)-\sqrt{d_1a_{33}}\left(N_3-\bar{N}_3\right)\right\}^2+\left(2\sqrt{d_1a_{11}a_{33}}-a_{13}\right)\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)\right]\right.$$
$$\left.+d_2\left[-a_{44}\left(N_4-\bar{N}_4\right)^2+\left(a_{43}-\frac{a_{34}d_1}{d_2}\right)\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right)\right]\right\} \quad (16)$$

The constants d_1 and d_2 as so chosen that, the coefficients of $\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)$ and $\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right)$ in (16) vanish.

Then we have $d_1=\frac{a_{13}^2}{4a_{11}a_{33}}>0$ and

$d_2=\frac{a_{13}^2a_{34}}{4a_{11}a_{33}a_{43}}>0$

With this choice of a constants d_1 and d_2.

$$\frac{dV}{dt}=-\left[\left\{\sqrt{a_{11}}\left(N_1-\bar{N}_1\right)-\frac{a_{13}}{2\sqrt{a_{11}}}\left(N_3-\bar{N}_3\right)\right\}^2+\frac{a_{13}^2a_{34}a_{44}}{4a_{11}a_{33}a_{43}}\left(N_4-\bar{N}_4\right)^2\right] \quad (17)$$

which is negative definite.

Hence, $E_{13}\left(\bar{N}_1,0,\bar{N}_3,\bar{N}_4\right)$ is globally asymptotically stable.

Theorem 4. *The equilibrium point $E_{14}\left(\bar{N}_1,\bar{N}_2,0,\bar{N}_4\right)$ is globally asymptotically stable.*

Proof : Let us consider the following Liapunov's function

$$V(N_1,N_2,N_4)=N_1-\bar{N}_1-\bar{N}_1\ln\left(\frac{N_1}{\bar{N}_1}\right)+d_1\left[N_2-\bar{N}_2-\bar{N}_2\ln\left(\frac{N_2}{\bar{N}_2}\right)\right]$$
$$+d_2\left[N_4-\bar{N}_4-\bar{N}_4\ln\left(\frac{N_4}{\bar{N}_4}\right)\right] \quad (18)$$

where d_1 and d_2 are suitable constants to be determined as in the subsequent steps.

The time derivative of V, along with solutions of (1), (2) and (4) can be written as

$$\frac{dV}{dt}=\left(\frac{N_1-\bar{N}_1}{N_1}\right)\frac{dN_1}{dt}+d_1\left(\frac{N_2-\bar{N}_2}{N_2}\right)\frac{dN_2}{dt}+d_2\left(\frac{N_4-\bar{N}_4}{N_4}\right)\frac{dN_4}{dt} \quad (19)$$

$$=\left(N_1-\bar{N}_1\right)\left(a_1-a_{11}N_1-a_{12}N_2\right)+d_1\left(N_2-\bar{N}_2\right)\left(a_2-a_{22}N_2+a_{21}N_1+a_{24}N_4\right)$$
$$+d_2\left(N_4-\bar{N}_4\right)\left(a_4-a_{44}N_4\right)$$

$$=\left(N_1-\bar{N}_1\right)\left(a_{11}\bar{N}_1+a_{12}\bar{N}_2-a_{11}N_1-a_{12}N_2\right)$$

$$+d_1\left(N_2-\bar{N}_2\right)\left(a_{22}\bar{N}_2-a_{21}\bar{N}_1-a_{24}\bar{N}_4-a_{22}N_2+a_{21}N_1+a_{24}N_4\right)$$

$$+d_2\left(N_4-\bar{N}_4\right)\left(a_{44}\bar{N}_4-a_{44}N_4\right)$$

$$= -a_{11}\left(N_1-\bar{N}_1\right)^2 - a_{12}\left(N_1-\bar{N}_1\right)\left(N_2-\bar{N}_2\right)$$
$$+d_1\left[-a_{22}\left(N_2-\bar{N}_2\right)^2 + a_{21}\left(N_1-\bar{N}_1\right)\left(N_2-\bar{N}_2\right) + a_{24}\left(N_2-\bar{N}_2\right)\left(N_4-\bar{N}_4\right)\right]$$
$$+d_2\left[-a_{44}\left(N_4-\bar{N}_4\right)^2\right]$$

$$\left.\begin{aligned}\frac{dV}{dt} &= -a_{11}\left(N_1-\bar{N}_1\right)^2 - \left[\sqrt{a_{22}}\left(N_2-\bar{N}_2\right) - \sqrt{d_2 a_{44}}\left(N_4-\bar{N}_4\right)\right]^2 \\ &\quad -\left(2\sqrt{d_2 a_{22} a_{44}} - a_{24}\right)\left(N_2-\bar{N}_2\right)\left(N_4-\bar{N}_4\right)\end{aligned}\right\} \quad (20)$$

With $d_1 = \frac{a_{12}}{a_{21}}$ and the constant d_2 as so chosen that the coefficient of $\left(N_2-\bar{N}_2\right)\left(N_4-\bar{N}_4\right)$ in (20) vanish.

Then we have $d_2 = \frac{a_{24}^2}{4a_{22}a_{44}} > 0$, with this choice of the constants d_1 and d_2

$$\frac{dV}{dt} = -\left[a_{11}\left(N_1-\bar{N}_1\right)^2 + \left\{\sqrt{a_{22}}\left(N_2-\bar{N}_2\right) - \frac{a_{24}}{2\sqrt{a_{22}}}\left(N_4-\bar{N}_4\right)\right\}^2\right] \quad (21)$$

which is negative definite.

Hence, $E_{14}\left(\bar{N}_1, \bar{N}_2, 0, \bar{N}_4\right)$ is globally asymptotically stable.

Theorem 5. *The equilibrium point* $E_{16}\left(\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4\right)$ *is globally asymptotically stable.*

Proof : Let us consider the following Liapunov's function

$$\left.\begin{aligned}V\left(N_1, N_2, N_3, N_4\right) &= N_1 - \bar{N}_1 - \bar{N}_1 \ln\left(\frac{N_1}{\bar{N}_1}\right) + d_1\left[N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right)\right] \\ &\quad + d_2\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right] + d_3\left[N_4 - \bar{N}_4 - \bar{N}_4 \ln\left(\frac{N_4}{\bar{N}_4}\right)\right]\end{aligned}\right\} \quad (22)$$

where d_1, d_2 and d_3 are suitable constants to be determined as in the subsequent steps.

Now, the time derivative of V, along with solutions of (1), (2), (3) and (4) can be written as

$$\frac{dV}{dt} = \left(\frac{N_1-\bar{N}_1}{N_1}\right)\frac{dN_1}{dt} + d_1\left(\frac{N_2-\bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + d_2\left(\frac{N_3-\bar{N}_3}{N_3}\right)\frac{dN_3}{dt} + d_3\left(\frac{N_4-\bar{N}_4}{N_4}\right)\frac{dN_4}{dt} \quad (23)$$

$$= \left(N_1-\bar{N}_1\right)\left(a_1 - a_{11}N_1 - a_{12}N_2 + a_{13}N_3\right) + d_1\left(N_2-\bar{N}_2\right)\left(a_2 - a_{22}N_2 + a_{21}N_1 + a_{24}N_4\right)$$

$$+d_2\left(N_3-\bar{N}_3\right)\left(a_3 - a_{33}N_3 - a_{34}N_4\right) + d_3\left(N_4-\bar{N}_4\right)\left(a_4 - a_{44}N_4 + a_{43}N_3\right)$$

$$= \left(N_1-\bar{N}_1\right)\left(a_{11}\bar{N}_1 + a_{12}\bar{N}_2 - a_{13}\bar{N}_3 - a_{11}N_1 - a_{12}N_2 + a_{13}N_3\right)$$

$$+d_1\left(N_2-\bar{N}_2\right)\left(a_{22}\bar{N}_2 - a_{21}\bar{N}_1 - a_{24}\bar{N}_4 - a_{22}N_2 + a_{21}N_1 + a_{24}N_4\right)$$

$$+d_2\left(N_3-\bar{N}_3\right)\left(a_{33}\bar{N}_3 + a_{34}\bar{N}_4 - a_{33}N_3 - a_{34}N_4\right)$$
$$+d_3\left(N_4-\bar{N}_4\right)\left(a_{44}\bar{N}_4 + a_{43}\bar{N}_3 - a_{44}N_4 + a_{43}N_3\right)$$

$$= a_{11}\left(N_1-\bar{N}_1\right)^2 - a_{12}\left(N_1-\bar{N}_1\right)\left(N_2-\bar{N}_2\right) + a_{13}\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)$$
$$+d_1\left[-a_{22}\left(N_2-\bar{N}_2\right)^2 + a_{21}\left(N_1-\bar{N}_1\right)\left(N_2-\bar{N}_2\right) + a_{24}\left(N_2-\bar{N}_2\right)\left(N_4-\bar{N}_4\right)\right]$$
$$+d_2\left[-a_{33}\left(N_3-\bar{N}_3\right)^2 - a_{34}\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right)\right]$$
$$+d_3\left[-a_{44}\left(N_4-\bar{N}_4\right)^2 + a_{43}\left(N_3-\bar{N}_3\right)\left(N_4-\bar{N}_4\right)\right]$$

$$\left.\begin{aligned}\frac{dV}{dt} &= -\left[\left\{\sqrt{a_{11}}\left(N_1-\bar{N}_1\right) - \sqrt{d_2 a_{33}}\left(N_3-\bar{N}_3\right)\right\}^2 + \left(2\sqrt{d_2 a_{11} a_{33}} - a_{13}\right)\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)\right] \\ &\quad -\left[\left\{\sqrt{d_1 a_{22}}\left(N_2-\bar{N}_2\right) - \sqrt{d_3 a_{44}}\left(N_4-\bar{N}_4\right)\right\}^2 + \left(2\sqrt{d_1 d_3 a_{22} a_{44}} - d_1 a_{24}\right)\left(N_2-\bar{N}_2\right)\left(N_4-\bar{N}_4\right)\right]\end{aligned}\right\} \quad (24)$$

With $d_1 = \frac{a_{12}}{a_{21}} > 0$ and the positive constants d_2, d_3 are so chosen that, the coefficients of $\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)$ and $\left(N_2-\bar{N}_2\right)\left(N_4-\bar{N}_4\right)$ in (24) vanish.

Then we have $d_2 = \dfrac{a_{13}^2}{4a_{11}a_{33}} > 0$ and

$$d_3 = \frac{a_{13}^2 a_{34}}{4a_{11}a_{33}a_{43}} > 0$$

With this choice of the constants d_1, d_2, d_3

$$\frac{dV}{dt} = -\left[\left\{\sqrt{a_{11}}\left(N_1-\bar{N}_1\right)-\frac{a_{13}}{2\sqrt{a_{11}}}\left(N_3-\bar{N}_3\right)\right\}^2+\left\{\sqrt{\frac{a_{21}a_{22}}{a_{11}}}\left(N_2-\bar{N}_2\right)-\frac{a_{24}}{2}\sqrt{\frac{a_{34}a_{44}}{a_{11}a_{33}a_{43}}}\left(N_4-\bar{N}_4\right)\right\}^2\right]$$

(25)

which is negative definite, when $a_{13}^2 a_{21} a_{22} a_{34} a_{44} = a_{24}^2 a_{11} a_{12} a_{33} a_{43}$

Hence, the normal steady state is globally asymptotically stable.

A NUMERICAL SOLUTION OF THE GROWTH RATE EQUATIONS

The numerical solutions of the growth rate basic equations (1), (2), (3), (4) have been computed employing the fourth order Runge-Kutta method. Some specific typically chosen values of system parameters characterizing in ecological model under investigation and properly chosen initial conditions. Making use of Matlab facility. What follows are the results of numerical computation and these are illustrated in figures (2) to (13) and some observations made here under.

Case(i): If $N_{i0} < \dfrac{K_i}{2}$, i = 1,2,3,4.

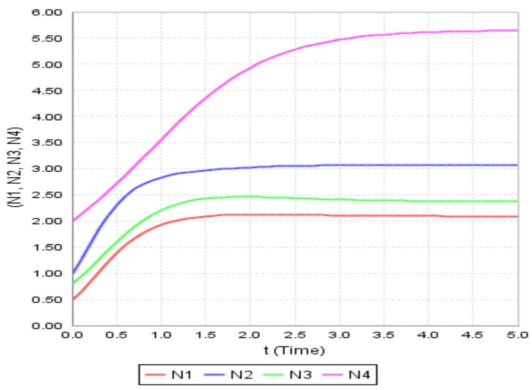

Fig. 2 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=3.6, a_2=4, a_3=2.7, a_4=1, a_{12}=0.18, a_{13}=0.98, a_{21}=0.2, a_{24}=0.09, a_{34}=0.1, a_{43}=0.15, K_1=1.4, K_2=2.5, K_3 =3, K_4 =4.2, N_{10}=0.5, N_{20}=1, N_{30}=0.8, N_{40}=2

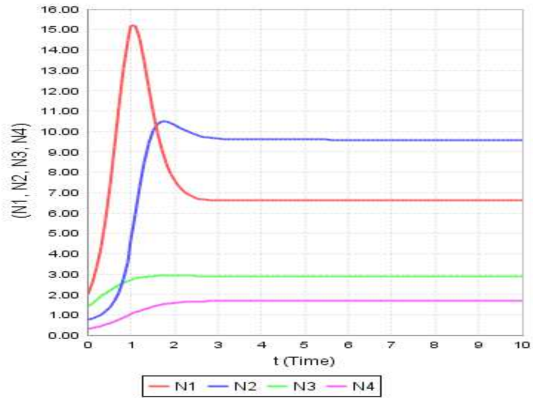

Fig. 3 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=0.88, a_2=0.48, a_3=2.1, a_4=1.4, a_{12}=0.45, a_{13}=1.65, a_{21}=0.18, a_{24}=0.72, a_{34}=0.22, a_{43}=0.33, K_1=4.4, K_2=1.6, K_3 =3.5, K_4 =1, N_{10}=2, N_{20}=0.75, N_{30}=1.4, N_{40}=0.3

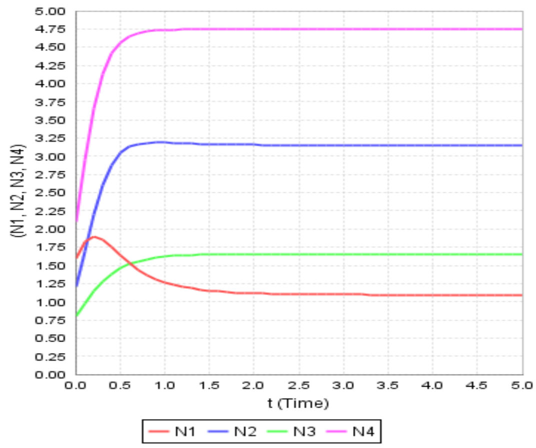

Fig. 4 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=5.7, a_2=5.6, a_3=4.4, a_4=7.2, a_{12}=1.36, a_{13}=0.14, a_{21}=0.34, a_{24}=0.07, a_{34}=0.16, a_{43}=0.24, K_1=3.8, K_2=2.8, K_3 =2, K_4 =4.5, N_{10}=1.6, N_{20}=1.2, N_{30}=0.8, N_{40}=2.1

Case (ii): If $N_{i0} > K_i$, i = 1,2,3,4.

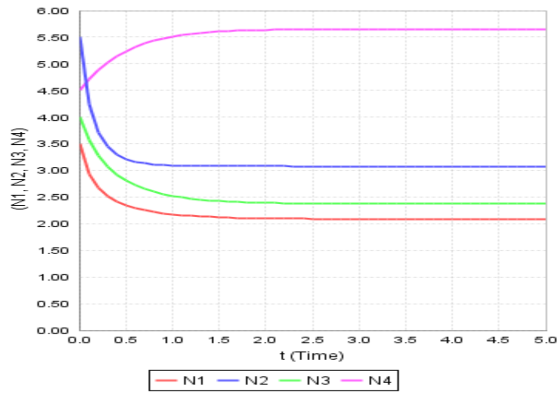

Fig. 5 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=3.6, a_2=4, a_3=2.7, a_4=1, a_{12}=0.18, a_{13}=0.98, a_{21}=0.2, a_{24}=0.09, a_{34}=0.1, a_{43}=0.15, K_1=1.4, K_2=2.5, K_3 =3, K_4 =4.2, N_{10}=3.5, N_{20}=5.5, N_{30}=4, N_{40}=4.5

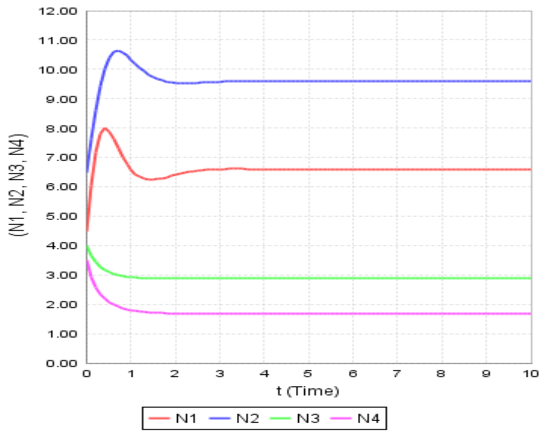

Fig. 6 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=0.88, a_2=0.48, a_3=2.1, a_4=1.4, a_{12}=0.45, a_{13}=1.65, a_{21}=0.18, a_{24}=0.72, a_{34}=0.22, a_{43}=0.33, K_1=4.4, K_2=1.6, K_3 =3.5, K_4 =1, N_{10}=4.5, N_{20}=6.5, N_{30}=4, N_{40}=3.5

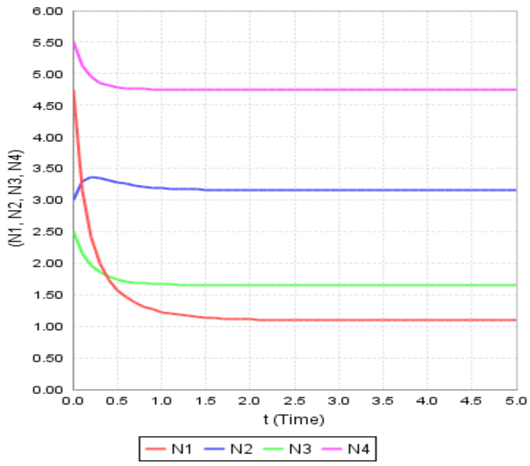

Fig. 7 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=5.7, a_2=5.6, a_3=4.4, a_4=7.2, a_{12}=1.36, a_{13}=0.14, a_{21}=0.34, a_{24}=0.07, a_{34}=0.16, a_{43}=0.24, K_1=3.8, K_2=2.8, K_3 =2, K_4 =4.5, N_{10}=4.75, N_{20}=3, N_{30}=2.5, N_{40}=6.5

Case (iii): If $\frac{K_i}{2} < N_{i0} < K_i$, i = 1,2,3,4.

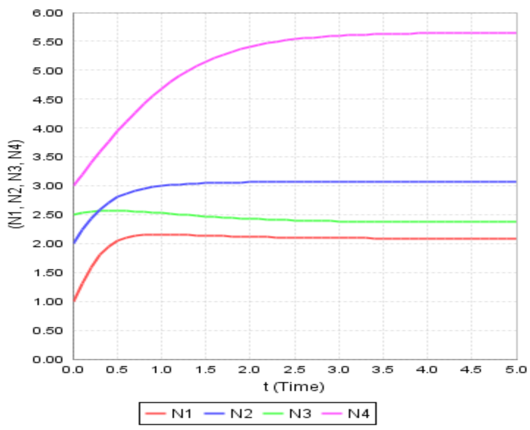

Fig. 8 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=3.6, a_2=4, a_3=2.7, a_4=1, a_{12}=0.18, a_{13}=0.98, a_{21}=0.2, a_{24}=0.09, a_{34}=0.1, a_{43}=0.15 K_1=1.4, K_2=2.5, K_3 =3, K_4 =4.2, N_{10}=1, N_{20}=2, N_{30}=2.5, N_{40}=3

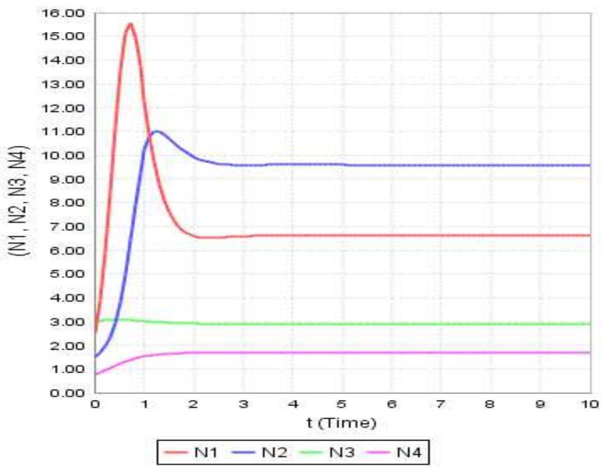

Fig. 9 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=0.88, a_2=0.48, a_3=2.1, a_4=1.4, a_{12}=0.45, a_{13}=1.65, a_{21}=0.18, a_{24}=0.72, a_{34}=0.22, a_{43}=0.33, K_1=4.4, K_2=1.6, K_3 =3.5, K_4 =1, N_{10}=2.5, N_{20}=1.5, N_{30}=3, N_{40}=0.75

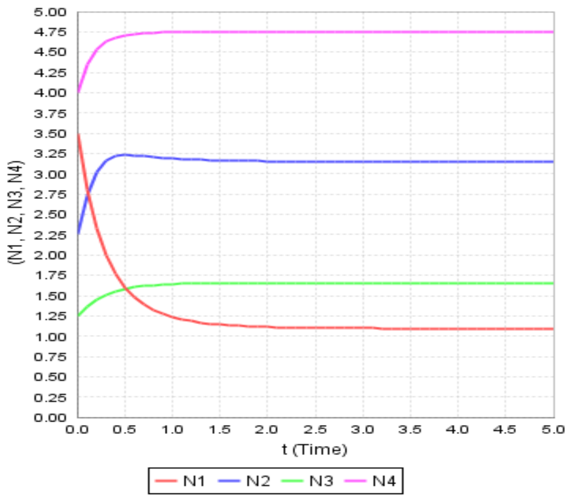

Fig. 10 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=5.7, a_2=5.6, a_3=4.4, a_4=7.2, a_{12}=1.36, a_{13}=0.14, a_{21}=0.34, a_{24}=0.07, a_{34}=0.16, a_{43}=0.24, K_1=3.8, K_2=2.8, K_3 =2, K_4 =4.5, N_{10}=3.5, N_{20}=2.25, N_{30}=1.25, N_{40}=4

Case (iv): If $N_{i0} = K_i$, i = 1,2,3,4.

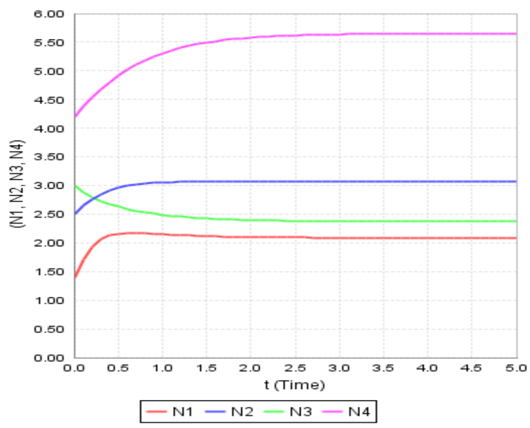

Fig. 11 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=3.6, a_2=4, a_3=2.7, a_4=1, a_{12}=0.18, a_{13}=0.98, a_{21}=0.2,

a_{24}=0.09, a_{34}=0.1, a_{43}=0.15 K_1=1.4, K_2=2.5, K_3 =3, K_4 =4.2, N_{10}=1.4, N_{20}=2.5, N_{30}=3, N_{40}=4.2

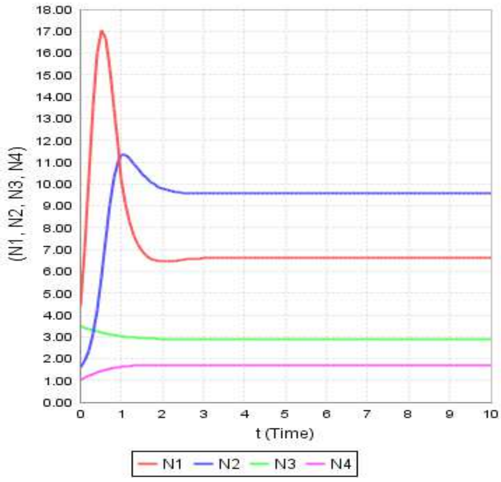

Fig. 12 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=0.88, a_2=0.48,a_3=2.1, a_4=1.4, a_{12}=0.45, a_{13}=1.65, a_{21}=0.18, a_{24}=0.72, a_{34}=0.22, a_{43}=0.33, K_1=4.4, K_2=1.6, K_3 =3.5, K_4 =1, N_{10}=4.4, N_{20}=1.6, N_{30}=3.5, N_{40}=1

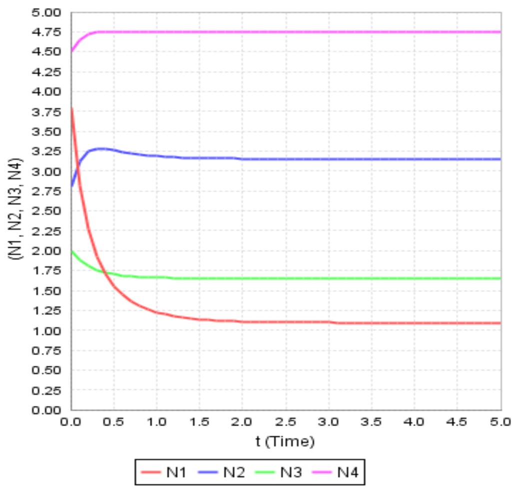

Fig. 13 Variation of N_1, N_2, N_3 and N_4 against time (t) for a_1=5.7, a_2=5.6, a_3=4.4, a_4=7.2, a_{12}=1.36, a_{13}=0.14, a_{21}=0.34, a_{24}=0.07, a_{34}=0.16, a_{43}=0.24, K_1=3.8, K_2=2.8, K_3 =2, K_4 =4.5, N_{10}=3.8, N_{20}=2.8, N_{30}=2, N_{40}=4.5

OBSERVATIONS OF THE ABOVE GRAPHS

Situation 1: This is a situation in which the carrying capacity of S_4 is highest and the forth species has the least natural birth rate. It is noticed that the first species is dominated by the third which itself dominated by the second and forth as shown in Fig. 2.

Situation 2: In this situation the second species is dominated by the third up to the time instant $t^* = 0.9$ and the second up to $t^* = 1.5$ after which the dominances are reversed. This is a situation in which the carrying capacity of S_4 is lowest. Further we noticed that the first and second species have a steep rise initially and then suffers a fall. (Fig. 3).

Situation 3: In this situation the first species dominates over the second and third till the time instant $t^* = 0.18$ and $t^* = 0.65$, which is reversed later. Further we notice that the forth species is with high natural birth rate. This is a case in which the commensal coefficient a_{24} is lowest. (Fig. 4).

Situation 4: In this situation initially the S_4 is dominated by the S_2 up to the time $t^* = 0.1$ and the dominance is reversed. Further the initial value of S_2 is highest and the forth species increase initially while the other three decrease. (Fig. 5).

Situation 5: In this situation the forth species is dominated by the third which itself dominated by the second and first. This is a case in which the commensal coefficient a_{13} is highest. Further we noticed that the first and second species have a steep rise initially and then suffers a fall. (Fig. 6).

Situation 6: In this situation initially the first species dominates over the second and third till the time instant $t^* = 0.17$ and $t^* = 0.4$ after which the dominance is reversed. Further we notice that the natural birth rates of first and second species are almost equal and all the four species decrease initially. (Fig. 7).

Situation 7: In this situation the S_2 is dominated by S_3 up to the time instant $t^* = 0.29$ after which dominate time we find reversal of the dominance. This is a case in which commensal coefficient a_{13} is highest. Further we notice that the third species has low growth rate. (Fig. 8).

Situation 8: This is a situation in which the initial value of the S_4 is lowest. It is notice that the forth species has low growth rate and the third is a weak competitor with no appreciable growth even from the start. (Fig. 9).

Situation 9: In this situation the initial value of the S_4 is highest. Initially the first species dominates over the second and third till the time instant $t^* = 0.12$ and $t^* = 0.5$ after which the dominance is reversed. Further we notice that only the first species decrease initially. (Fig. 10).

Situation 10: This is a situation at the carrying capacity of the S_4 is highest. Initially the second species is dominated by the third species up to the time instant $t^* = 0.2$ after which the dominance is reversed. Further we notice that the first species has the least initial value. (Fig. 11).

Situation 11: In this situation initially the S_2 is dominated by S_3 up to time instant $t^* = 0.16$ and the S_1 up to $t^* = 1$ after these dominate times we find reversal of the dominance. It is notice that the forth species has low growth rate and the third is a weak competitor with no appreciable growth even from the start. (Fig. 12).

Situation 12: In this situation the third species has the least natural birth rate. Initially the second and third species are dominated by the first up to the time instant $t^* = 0.1$ and $t^* = 0.4$ after which the dominance is reversed. It is notice that the forth species has low growth rate. Further we notice that the first and third species decrease initially. (Fig. 13).

CONCLUSION

Investigate some relation-chains between the species such as Prey-Predation, Neutralism, Commensalism, Mutualism, Competition and Ammensalism between four species (S_1, S_2, S_3, S_4) with the population relations.

The present paper deals with an investigation on global stability of a Four Species Syn Eco-System with Commensal Prey-Predator pair with Prey-Predator pair of Hosts. The System comprises of a Prey (S_1), a Predator (S_2) that survives upon S_1, two Hosts S_3 and S_4 for which S_1, S_2 are Commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 is Prey for S_4 and S_4 is Predator for S_3. The pair (S_1, S_2) may be referred as 1st level Prey-Predator and the pair (S_3, S_4) the 2nd level Prey-Predator. It is observed that, all the five equilibrium states E_7,E_{12},E_{13},E_{14},E_{16} are globally asymptotically stable. This is achieved by constructing suitable Liapunov's fuction. Furher the numarical solutions for the growth rate equations are computed using Runge-Kutta foutrh order method in four cases.

(i): The initial values of the three species are less than half the respective their carrying capacities.

(ii): The initial values of the three species are greater than their respective carrying capacities

(iii): The initial values of the three species are lie between half their respective carrying capacities and its carrying capacities.

(iv): The initial values of the three species are equal their respective carrying capacities.

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