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# QUASI – UMBILICAL HYPERSURFACES OF PARA – SASAKIAN MANIFOLDS

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**ABSTRACT:**

The present paper deal with the properties of quasi – umbilical hypersurfaces of para – Sasakian manifolds. Some theorems are obtained. Further, we have studied Q – quasi umbilical hypersurface with $(\tilde{\phi}, \tilde{g}, u, v, \lambda)$- structure of a para – Sasakian manifold.

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## 1. INTRODUCTION:

An n – dimensional differentiable manifold M is called an almost para - contact structure $(\phi, \xi, \eta)$ consisting of a (1, 1) tensor field $\phi$, a vector field $\xi$ and a 1 – form $\eta$ satisfying

(1.1) $\phi^2 = I - \eta \otimes \xi$, $\eta(\xi) = 1$, $\eta o \phi = 0$, $\phi\xi = 0$.

Let g be a compatible Riemannian metric with $(\phi, \xi, \eta)$- structure, such that

(1.2) $g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y)$

Or equivalently,

(1.3) $g(X, \phi Y) = g(\phi X, Y)$ and $g(X, \xi) = \eta(X)$

For all $X, Y \in TM$. Then M becomes an almost para – contact Riemannian manifold equipped with an almost para – contact Riemannian structure - $(\phi, \xi, \eta, g)$.

An almost para – contact Riemannian manifold is called a p – Sasakian manifold if it satisfies

(1.4) $(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)$; $X, Y \in TM$

Where $\nabla$ is a Levi – Civita connection of the Riemannian metric g.

From the above equation it follows that

(1.5) $\nabla_\xi X = \phi X$

(1.6) $(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X$ ; $X \in TM$ .

In an n – dimensional para – Sasakian manifold M, the curvature tensor R, the Ricci tensor S and the Ricci operator P satisfy

(1.7) $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$

(1.8) $R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi$

(1.9) $R(\xi, X)\xi = X - \eta(X)\xi$

(1.10) $S(X,\xi) = -(n-1)\eta(X)$

(1.11) $P\xi = -(n-1)\xi$

(1.12) $\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X)$

(1.13) $\eta(R(X,Y)\xi) = 0$

(1.14) $\eta(R(\xi, X)Y) = \eta(X)\eta(Y) - g(X,Y)$

An almost para – contact Riemannian manifold m is said to be $\eta$ - Einstein [2], if the Ricci operator P satisfies

(1.15) $P = aId + b\eta \otimes \xi$

Where a, b are smooth functions on the manifold. In particular, if b = 0, then M is an Einstein manifold.

Let (M, g) be an n –dimensional Riemannian manifold. Then the concircular curvature tensor C and the Weyl conformal curvature W are defined by [2]

(1.16) $C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}$

(1.17) $W(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}\{R(Y,Z)X - R(X,Z)Y + g(Y,Z)PX - g(X,Z)PY\}$

$$+\frac{r}{(n-1)(n-2)}\{g(Y,Z)X-g(X,Z)Y\}$$

For all $X,Y,Z\in TM$ , respectively, where r is the scalar curvature of M.

## 2. Hypersurface of para – Sasakian manifolds:

Let M be an n – dimensional Riemannian manifold with positive definite metric g and let $\tilde{M}$ be a hypersurface immersed in M. If i* denotes the differential of the immersion i of $\tilde{M}$ into M and $\tilde{X}$ is a vector field on M. Let N be the unit normal field to M. The induced metric $\tilde{g}$ on $\tilde{M}$ is defined by

(2.1) $\tilde{g}(\tilde{X},\tilde{Y})=g(\tilde{X},\tilde{Y})$

we have,

(2.2) $\tilde{g}(\tilde{X},N)=0$ and $\tilde{g}(N,N)=1$

If $\nabla$ is the Riemannian connection in M, then Gauss and Weingarten formula are given respectively by

(2.3) $\nabla_{\tilde{X}}\tilde{Y}=\tilde{\nabla}_{\tilde{X}}\tilde{Y}+h(\tilde{X},\tilde{Y})N$ ,

(2.4) $\nabla_{\tilde{X}}N=-H(\tilde{X})$

Where $\tilde{\nabla}$ is the induced Riemannian connection in $\tilde{M}$ and h is the second fundamental tensor satisfying

(2.5) $h(\tilde{X},\tilde{Y})=h(\tilde{X},\tilde{Y})=g(H(\tilde{X}),\tilde{Y})$.

**Remark:** *On all objects of $\tilde{M}$ will be denoted with hyphen ' $\tilde{\cdot}$ ' placed over them , e.g. $\tilde{\phi},\tilde{X}$ etc.*

Now suppose that $(\phi,\xi,\eta,g)$ is para – Sasakian structure on M. Then every vector field X on M is decomposed as

(2.6) $X=\tilde{X}+\omega(\tilde{X})N$

Where $\omega$ is 1 – form on M, and for every vector field $\tilde{X}$ on $\tilde{M}$ and the normal N, we have

(2.7) $\phi\tilde{X}=\tilde{\phi}\tilde{X}+u(\tilde{X})N$

(2.8) $\phi N=-\tilde{U}$

(2.9) $\xi=\tilde{V}+\lambda N$

(2.10) $\eta(\tilde{X}) = v(\tilde{X})$

Where $\tilde{\phi}$ is a (1, 1) type tensor; $\tilde{U}, \tilde{V}$ are vector fields; u, v are 1 – form on $\tilde{M}$ and $\lambda$ is a scalar function on $\tilde{M}$ .

If $u \neq 0$ , we call $\tilde{M}$ a non - invariant hypersurface of M [2].

From (2.7), (2.8), (2.9) and (2.10); we have

(2.11) (a) $\tilde{\phi}^2 \tilde{X} = -\tilde{X} + u(\tilde{X})\tilde{U} + v(\tilde{X})\tilde{V}$

(b) $u(\tilde{\phi}\tilde{X}) = \lambda v(\tilde{X}), \; v(\tilde{\phi}\tilde{X}) = -\eta(N)u(\tilde{X})$

(c) $\tilde{\phi}\tilde{U} = -\eta(N)\tilde{V}, \; \tilde{\phi}\tilde{V} = \lambda\tilde{U}$

(d) $u(\tilde{U}) = 1 - \lambda\eta(N), \; u(\tilde{V}) = 0$

(e) $v(\tilde{U}) = 0, \; v(\tilde{V}) = 1 - \lambda\eta(N)$

And the induced metric $\tilde{g}$ on $\tilde{M}$ is given by

(2.12) $g(\tilde{\phi}\tilde{X}, \tilde{\phi}\tilde{Y}) = g(\tilde{X}, \tilde{Y}) - u(\tilde{X})u(\tilde{Y}) - v(\tilde{X})v(\tilde{Y})$

Where $\tilde{g}(\tilde{U}, \tilde{X}) = u(\tilde{X})$ , $\tilde{g}(\tilde{V}, \tilde{X}) = v(\tilde{X})$.

If we consider $\eta(N) = \lambda$ , we get following structures on $\tilde{M}$

(2.13) (a) $\tilde{\phi}^2 = -Id + u \otimes \tilde{U} + v \otimes) \tilde{V}$

(b) $\tilde{\phi}\tilde{U} = -\lambda\tilde{V}, \; \tilde{\phi}\tilde{V} = \lambda\tilde{U}$

(c) $uo\tilde{\phi} = \lambda v, \; vo\tilde{\phi} = -\lambda u$

(d) $u(\tilde{U}) = 1 - \lambda^2, \; u(\tilde{V}) = 0$

(e) $v(\tilde{U}) = 0, \; v(\tilde{V}) = 1 - \lambda^2$

A manifold $\tilde{M}$ with a metric $\tilde{g}$ satisfying (2.12) and (2.13) is called manifold with $(\tilde{\phi}, \tilde{g}, u, v, \lambda)$- structure [4].

Differentiating (2.7), (2.8), (2.9) and (2.10) covariantly and using (1.5), (1.6), (2.3) and (2.4), we can state the following theorem:

**Theorem 2.1** Let $\tilde{M}$ be the hypersurface with $(\tilde{\phi}, \tilde{g}, u, v, \lambda)$ - structure of a para – Sasakian manifold M, then following relation hold:

(2.14) $\left(\tilde{\nabla}_{\tilde{Y}}\phi\right)\left(\tilde{X}\right) = 2v\left(\tilde{X}\right)v\left(\tilde{Y}\right) - v\left(\tilde{X}\right)\tilde{Y} - g\left(\tilde{X},\tilde{Y}\right)\tilde{V} + u\left(\tilde{X}\right)H\tilde{Y} - h\left(\tilde{X},\tilde{Y}\right)\tilde{U}$

(2.15) $\left(\tilde{\nabla}_{\tilde{Y}}u\right)\left(\tilde{X}\right) = -h\left(\tilde{Y},\tilde{\phi}\tilde{X}\right) - \lambda g\left(\tilde{X},\tilde{Y}\right)$

(2.16) $\tilde{\nabla}_{\tilde{Y}}\tilde{U} = \lambda\tilde{Y} + 2\lambda v\left(\tilde{Y}\right) - \tilde{\phi}H\tilde{Y}$

(2.17) $\tilde{\nabla}_{\tilde{Y}}\tilde{V} = \tilde{\phi}\tilde{Y} + \lambda H\tilde{Y}$

(2.18) $h\left(\tilde{Y},\tilde{V}\right) = u\left(\tilde{Y}\right)$

(2.19) $h\left(\tilde{Y},\tilde{U}\right) = u\left(H\tilde{Y}\right)$

Since, $g(HY, X) = h(Y, X)$, then from (2.12) and (2.19) we can state the following theorem as a corollary of theorem 2.1:

**Theorem 2.2** Let $\tilde{M}$ be the hypersurface with $(\tilde{\phi}, \tilde{g}, u, v, \lambda)$ - structure of a para – Sasakian manifold M, then we have

$$h\left(\tilde{Y},\tilde{U}\right) \neq 0 \Rightarrow H\tilde{U} = 0.$$

## 3. Quasi umbilical hypersurface:

If the second fundamental form given by

(3.1) $h(X,Y) = \alpha g(X,Y) + \beta q(X)q(Y)$

Where $\alpha, \beta$ are scalar functions and q is 1 – form then $\tilde{M}$ is called quasi umbilical hypersurface and if $g(Q, X) = q(X)$, where Q is a vector field, then $\tilde{M}$ is called Q – quasi umbilical hypersurface.

If $\alpha = 0, \beta \neq 0$ then Q – quasi umbilical hypersurface $\tilde{M}$ is called cylindrical hypersurface[3].

If $\alpha \neq 0, \beta = 0$ then Q – quasi umbilical hypersurface is totally geodesic.

Using (3.1) in theorem 2.1, we have (3.2)

$$\left(\tilde{\nabla}_{\tilde{Y}}\phi\right)\left(\tilde{X}\right)=2v\left(\tilde{X}\right)v\left(\tilde{Y}\right)-v\left(\tilde{X}\right)\tilde{Y}-g\left(\tilde{X},\tilde{Y}\right)\tilde{V}+u\left(\tilde{X}\right)\left\{\alpha\tilde{Y}+\beta q\left(\tilde{Y}\right)Q\right\}-\left\{\alpha g\left(\tilde{X},\tilde{Y}\right)+\beta q\left(\tilde{X}\right)q\left(\tilde{Y}\right)\right\}\tilde{U} \quad (3.3)$$

$$\left(\tilde{\nabla}_{\tilde{Y}}u\right)\left(\tilde{X}\right)=-\left\{\alpha g\left(\tilde{Y},\tilde{\phi}\tilde{X}\right)+\beta q\left(\tilde{\phi}\tilde{X}\right)q\left(\tilde{Y}\right)\right\}-\lambda g\left(\tilde{X},\tilde{Y}\right)$$

(3.4) $$\tilde{\nabla}_{\tilde{Y}}\tilde{U}=\lambda\tilde{Y}+2\lambda v\left(\tilde{Y}\right)-\tilde{\phi}\left\{\alpha\tilde{Y}+\beta q\left(\tilde{Y}\right)Q\right\}$$

(3.5) $$\tilde{\nabla}_{\tilde{Y}}\tilde{V}=\tilde{\phi}\tilde{Y}+\lambda\left\{\alpha\tilde{Y}+\beta q\left(\tilde{Y}\right)Q\right\}$$

Put $\tilde{Y}=\tilde{U}$ in (3.1), we have

$$h\left(\tilde{X},\tilde{U}\right)=\alpha u\left(\tilde{X}\right)+\beta q\left(\tilde{X}\right)q\left(\tilde{U}\right)$$

Using theorem (2.2), we get

(3.6) $$\alpha u\left(\tilde{X}\right)+\beta q\left(\tilde{X}\right)q\left(\tilde{U}\right)\neq 0$$

Put X = U, then $\alpha u\left(\tilde{U}\right)+\beta q\left(\tilde{U}\right)q\left(\tilde{U}\right)\neq 0$

(3.7) $$\left|q\left(\tilde{U}\right)\right|^{2}\neq\frac{-\alpha}{\beta}\left(1-\lambda^{2}\right)$$

If we take $\tilde{X}=\tilde{V}$ in (3.6), we get $\beta q\left(\tilde{V}\right)q\left(\tilde{U}\right)\neq 0$

Since $\beta\neq 0, q\left(\tilde{U}\right)\neq 0\Rightarrow q\left(\tilde{V}\right)\neq 0$, therefore $q\left(\tilde{V}\right)=v(Q)\neq 0$.

This leads to the following theorem:

Theorem 3.1 On the Q – quasi umbilical hypersurface $\tilde{M}$ with $\left(\tilde{\phi},\tilde{g},u,v,\lambda\right)$ - structure of a para Sasakian manifold M, we have

(3.8) $$q\left(\tilde{V}\right)=v(Q)\neq 0,\ \left|q\left(\tilde{U}\right)\right|^{2}\neq\frac{-\alpha}{\beta}\left(1-\lambda^{2}\right).$$

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