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THE UPPER CONNECTED EDGE MONOPHONIC NUMBER OF A GRAPH

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ABSTRACT

A connected edge monophonic set M in a connected graph $G = (V, E)$ is called a *minimal connected edge monophonic set* if no proper subset of M is a connected edge monophonic set of G. The *upper connected edge monophonic number* $m_{1c}^+(G)$ is the maximum cardinality of a *minimal connected edge monophonic set* of G. Connected graphs of order p with upper connected edge monophonic number 2 and p are characterized. It is shown that for any positive integers $2 \le a < b \le c$, there exists a connected graph G with $m_1(G) = a$, $m_{1c}(G) = b$ and $m_{1c}^+(G)$=c, where $m_1(G)$ is the monophonic number and $m_{1c}(G)$ is the connected Edge monophonic number of a graph G.

***Keywords*:** monophonic number, edge monophonic number, connected edge monophonic number, upper connected edge monophonic number.

AMS Subject Classification: 05*C*05.

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [1].
$N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the neighborhood of the vertex v in G. For any set M of vertices of G, the *induced subgraph* $\langle M \rangle$ is the maximal subgraph of G with vertex set M. A vertex v is an *extreme vertex* of a graph G if $\langle N(v) \rangle$ is complete. *A chord* of a path $u_0, u_1, u_2, \ldots, u_h$ is an edge u_iu_j with $j \ge i + 2$. An u-v path is called a *monophonic path* if it is a chordless path. For two vertices u and v in a connected graph G,

the *monophonic distance* $d_m(u,v)$ is the length of the longest $u-v$ monophonic path in *G*.

An $u-v$ monophonic path of length $d_m(u,v)$ is called an $u-v$ *monophonic*. For a vertex *v* of *G*, the *monophonic eccentricity* $e_m(v)$ is the monophonic distance between *v* and a vertex farthest from *v*. The minimum monophonic eccentricity among the vertices in the *monophonic radius*, $rad_m(G)$ and the maximum monophonic eccentricity is the *monophonic diameter* $diam_m(G)$ of *G*. A *monophonic set* of *G* is a set $M \subseteq V(G)$ such that every vertex of *G* is contained in a monophonic path joining some pair of vertices in *M*. The *monophonic number* $m(G)$ of *G* is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* of *G*. The monophonic number of a graph *G* is studied in [2,3,4].An *edge monophonic set* of *G* is a set $M \subseteq V(G)$ such that every edge of *G* is contained in a monophonic path joining some pair of vertices in *M*. The *edge monophonic number* $m_1(G)$ of *G* is the minimum order of its edge monophonic sets and any edge monophonic set of order $m_1(G)$ is a *minimum edge monophonic set* of *G*. The edge monophonic number of a graph *G* is introduced in [7] and further studied in [6]. A *connected edge monophonic set* of a graph *G* is an edge monophonic set *M* such that the subgraph <*M*> induced by *M* is connected. The minimum cardinality of a connected edge monophonic set of *G* is the *connected edge monophonic number* of *G* and is denoted by $m_{1c}(G)$. A connected edge monophonic set of cardinality $m_{1c}(G)$ is called a *m_{1c}-set* of *G* or a *minimum connected edge monophonic set* of *G*. The connected edge monophonic number of a graph is studied in [5].
The following theorems are used in the sequel.

Corollary1.1[6] Each simplicial vertex of *G* belongs to every edge monophonic set of *G*.

Corollary 1.2.[6] For any non trivial tree *T*, the edge monophonic number $m_1(G)$ equals the number of end vertices in *T*. In fact, the set of all end vertices of *T* is the unique minimum edge monophonic set of *T*.

Theorem 1.3.[5] Each semi-simplicial vertex of a graph G belongs to every connected edge monophonic set of G.

Corollary 1.4.[5] Each simplicial vertex of a graph G belongs to every connected edge monophonic set of G.

Theorem 1.5.[5] Let G be a connected graph, v be a cut vertex of G and let M be a connected edge monophonic set of G. Then every component of $G-v$ contains an element of M.

Theorem 1.6.[5] Each cut vertex of a connected graph G belongs to every minimum connected edge monophonic set of G.

Corollary 1.7.[5]

i) For any non-trivial tree T of order $p, m_{1c}(T)=p$.

ii) For the complete graph $K_p (p \geq 2), m_{1c}(K_p)=p$.

2 The Upper Connected Edge Monophonic Number of a Graph.

Definition 2.1. A connected edge monophonic set M in a connected graph G is called a minimal connected edge monophonic set if no proper sub set of M is a connected edge monophonic set of G. The upper connected edge monophonic number

$m_{1c}^{+}(G)$ is the maximum cardinality of a minimal connected edge monophonic set of G.

Example 2.2. For the graph G given in Figure 2.1, $M_1 = \{v_1, v_2, v_3, v_4\}$, $M_2 = \{v_1, v_2, v_3, v_5\}$, $M_3 = \{v_1, v_2, v_3, v_6\}$and $M_4 = \{v_1, v_2, v_3, v_7\}$ are minimum connected edge monophonic sets of G so that $M_{1c}(G) = 4$. The sets $M' = \{v_1, v_4, v_5, v_6, v_7\}$, are also connected edge monophonic sets of G and it is clear that no proper subsets of M', M'' and M''' are connected edge monophonic set so that M', M'' and M''' are minimal edge monophonic sets of G. It is easily verified that there is no minimal connected edge monophonic set M with $|M| \geq 5$. Hence it follows that $m_{1c}^{+}(G) = 4$.

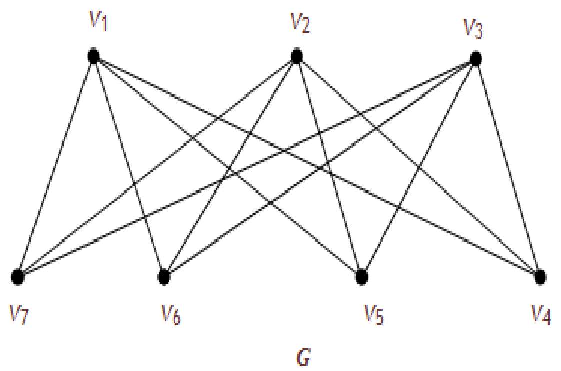

Figure 2.1

Remark 2.3. Every minimum connected edge monophonic set of G is a minimal connected edge monophonic set of G. The converse is not true. For the graph G given in Figure 2.2, $M' = \{v_1, v_4, v_5, v_6, v_7\}$ is a minimal connected edge monophonic set and is not a minimum connected edge monophonic set of G.

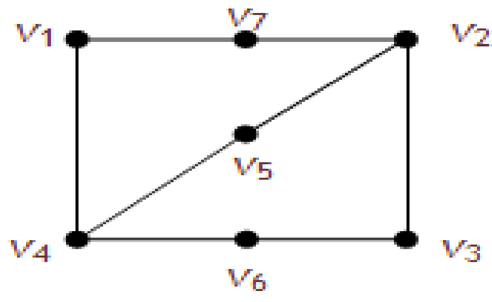

Figure 2.2

Theorem 2.4. For any connected graph G, $2 \leq m_{1c}(G) \leq m_{1c}^{+}(G) \leq p$.

Proof. Any connected edge monophonic set need at least two vertices and so $m_c(G) \geq 2$. Since every minimum connected edge monophonic set is a minimal connected edge monophonic set, $m_{1c}(G) \leq m_{1c}^{+}(G)$. Also, Since $V(G)$ induces a connected edge monophonic set of G, it is clear that $m_{1c} \leq p$. Thus $2 \leq m_c(G) \leq m_c^{+}(G) \leq p$.

Remark 2.5. For the graph $K_2, m_{1c}(K_2) = 2$. For any non-trivial tree T of order $p, m_{1c}^{+}(T) = p$. Also, all the in equalities in Theorem 2.4, are strict. For the graph G given in Figure 2.2, $m_{1c}(G) = 3, m_{1c}^{+}(G) = 4,\ p = 6$ so that $2 < m_{1c}(G) < m_{1c}^{+}(G) < p$.

Theorem 2.6. For any connected graph $G, m_{1c}(G) = p$ if and only it $m_{1c}^{+}(G) = p$

Proof. Let $m_{1c}^{+}(G) = p$. Then $M = V(G)$ is the unique minimal edge monophonic set of G. Since no proper subset of M is a connected edge monophonic set, it is clear that M is the unique minimum connected edge monophonic set of G and so $m_{1c}(G) = p$. The converse follows from Theorem 2.4.

Theorem 2.7. Every simplicial vertex of a connected graph G belongs to every minimal connected edge monophonic set of G.

Proof. Since every minimal connected edge monophonic set is an edge monophonic set, the result follows from Corollary 1.1.

Theorem 2.8. Let G be a connected graph containing a cut-vertex v. Let M be a minimal connected edge monophonic set of G, then every component of $G - v$ contains an element of M.

Proof. Let v be a cut-vertex of G and M be a minimal connected edge monophonic set of G. Suppose there exists a component say G_1 of $G - v$ such that G_1 contains no vertex of M. By Theorem 2.7, M contains all simplicial vertices of G and hence it follows that G_1 does not contain any simplicial vertex of G. Thus G_1 contains at least one edge say xy. Since M is the minimal connected edge monophonic set, xy lies on the $u - w$ monophonic path $P: u, u_1, u_2, \ldots, v, \ldots, x, y, \ldots, v_1, \ldots, v, \ldots, w$. Since v is a cut-vertex of G, the $u - x$ and $y - w$ sub path of P both contains v and so P is not a path , which is a contradiction.

Theorem 2.9. Every cut-vertex of a connected graph G belongs to every minimal connected edge monophonic set of G.

Proof. Let u be any cut-vertex of G and let $G_1, G_2, \ldots, G_r (r > 2)$ be the components of $G - \{u\}$. Let M be any connected edge monophonic set of G. Then M contains at least element from each $G_i (1 \le i \le r)$ Since $G[M]$ is connected, it follows that $u \notin M$.

Corollary 2.10. For a connected graph G with k simplicial vertices and l cut-vertices, $m_{1c}^{+}(G) \ge \max\{2, k + l\}$.

Proof. This follows from Theorem2.7 and 2.9.

Corollary 2.11. For the complete graph $G = K_p, m_{1c}^{+}(G) = p$.

Proof. This is follows from Theorem 2.7.

Corollary 2.12. For any tree $T, m_{1c}^{+}(T) = p$.

Proof. This follows from Corollary 2.11.

REALISATION RESULTS

Theorem 2.13. For positive integers r_m, d_m and $l > d_m - r_m + 3$ with $r_m < d_m \le 2r_m$, there exists a connected graph G with $rad_m(G) = r_m, diam_m(G) = d_m$ and $m_{1c}^{+}(G) = l$.

Proof. When $r_m = 1$, we let $G = K_{1,l-1}$. Then the result follows from Corollary 5.30. Let $r_m \ge 2$, let $C_{r_m+2}: v_1, v_2, \ldots, v_{r_m+2}, v_1$ be a cycle of length $r_m + 2$ and let $P_{d_m-r_m+1}: u_0, u_1, u_2, \ldots, u_{d_m-r_m}$ be a path of length $d_{d_m-r_m+1}$. Let H be a graph obtained from C_{r_m+2} and $P_{d_m-r_m+1}$ by identifying v_1 in C_{r_m+2} and u_0 in $P_{d_m-r_m+1}$. Now add $l - d_m + r_m - 3$ new vertices $w_1, w_2, \ldots, w_{l-d_m+r_m-3}$ to H and join each $w_i\ (1 \le i < l - d_m + r_m - 3)$ to the vertex $u_{d_m-r_m-1}$ and obtain the graph G as shown in

Figure 2.3. Then $rad_m(G) = r_m$ and $diam_m(G) = d_m$.

$$M = \{u_0, u_1, u_2, ..., u_{d_m - r_m}, w_1, w_2, .., w_{l-d_m+r_m-3}\}$$

Let be the set of cut-vertices and end-vertices of *G*. By Corollary 1.4 and Theorem 1.6, *M* is a subset of every connected edge monophonic set of *G*. It is clear that *M* is not a connected edge monophonic set of *G*. Also $M \cup \{x\}$, where $x \notin M$ is not a connected edge monophonic set of *G*. However $M_1 = M \cup \{v_2, v_3\}$ is a connected edge monophonic set of *G*.

Now, we show that M_1 is a minimal connected edge monophonic set of *G*. Assume, to the contrary, that M_1 is not a minimal connected edge monophonic set of *G*. Then there is a proper subset *T* of M_1 such that *T* is connected edge monophonic set of *G*. Let $y \in M_1$ and $y \notin T$. By Theorem 1.3, $y \neq w_i (1 \leq i \leq l - d_m + r_m - 3$. Also by Theorem 1.6, $y \neq u_i$ $(1 \leq i \leq d_m - r_m)$. Then *T* is not a connected edge monophonic set of *G*, which is a contradiction. Thus, M_1 is a minimal connected edge monophonic set of *G* and so $m^+_{1c}(G) \geq l$. Let M' be a minimal connected edge monophonic set of *G* such that $|M'| > l$. By Theorems 1.3 and 1.5, M' contains *M*. Since, $M_1 = M \cup \{v_2, v_3\}$ or $M_2 = M \cup \{v_2, v_{r_m+2}\}$ or $M_3 = M \cup \{v_{r_m+1}, v_{r_m+2}\}$ is also a connected edge monophonic set of *G* and $< M' >$ is connected, it follows that M' contains either M_1 or M_2 or M_3, which is a contradiction to M' is a minimal connected edge monophonic set of G. Therefore $m^+_{1c}(G) = l$.

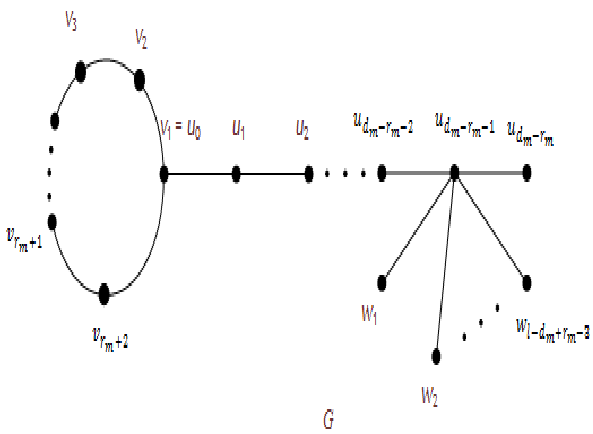

Figure 2.3

In view of Theorem 2.4, we have the following realisation result.

Theorem 2.14. For any positive integers $2 \leq a < b \leq c$, there exists a connected graph G such that $m_1(G) = a$, $m_{1c}(G) = b$ and $m^+_{1c}(G) = c$.

Proof. If $2 \leq a < b = c$, let G be any tree of order b with a end-vertices. Then by Corollary 1.2, $m_1(G) = a$, by Corollary 1.7(i), $m_{1c}(G) = b$ and by Corollary 2.12, $m^+_{1c}(G) = b$. Let $2 \leq a < b < c$. Now, we consider four cases.

Case 1. Let $a > 2$ and $b - a \geq 2$. Then $b - a + 2 \geq 4$, let $P_{b-a+2}: v_1, v_2, ..., v_{b-a+2}$ be a path of length $b - a + 1$. Add $c - b + a - 1$ new vertices $w_1, w_2, ..., w_{c-b}, u_1, u_2, ..., u_{a-1}$ to P_{b-a+2} and join $w_1, w_2, ..., w_{c-b}$ to both v_1 and v_3 and also join $u_1, u_2, ..., u_{a-1}$ to both v_1 and v_2, there by producing the graph G of Figure 2.4. Let $M = \{u_1, u_2, ..., u_{a-1}, v_{b-a+2}\}$ be the set of all simplicial vertices of G. By Corollary 1.1, every edge monophonic set of G contains M. It is clear that M is an edge monophonic set of G so that $m_1(G) = a$.

Let $M_1 = M \cup \{v_2, v_3, \dots, v_{b-a+1}\}$. By Corollary 1.4 and Theorem 1.6 each connected edge monophonic set contains M_1. It is clear that M_1 is a connected edge monophonic set of G so that $M_{1c}(G) = b$.

Let $M_2 = M_1 \cup \{w_1, w_2, \dots, w_{c-b}\}$. It is clear that M_2 is a connected edge monophonic set of G. Now, we show that M_2 is a minimal connected edge monophonic set of G.

Assume, to the contrary, that M_2 is not a minimal connected edge monophonic set. Then there is a proper subset T of M_2 such that T is a connected edge monophonic set of G. Let $v \in M_2$ and $v \notin T$. By Corollary 1.4 and Theorem 1.6 it is clear that $v = w_i$, for some $i = 1,2,\dots,c-b$.

Clearly, this w_i does not lie on a monophonic path joining any pair of vertices of T and so T is not a connected edge monophonic set of G, which is a contradiction. Thus M_2 is a minimal connected edge monophonic set of G and so $m_{1c}^{+}(G) \geq c$. Since the order of the graph is $c + 1$, it follows that $m_{1c}^{+}(G) = c$.

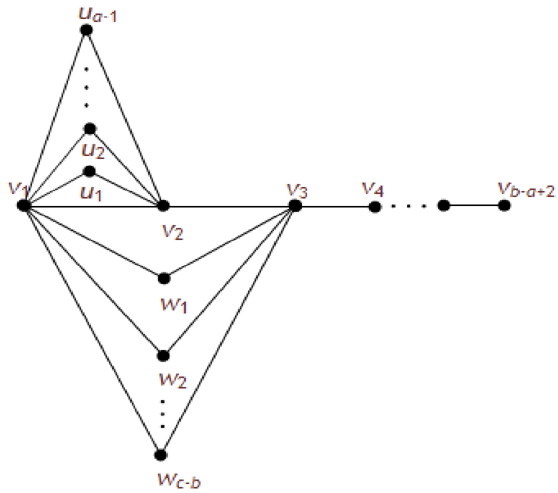

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Figure 2.4

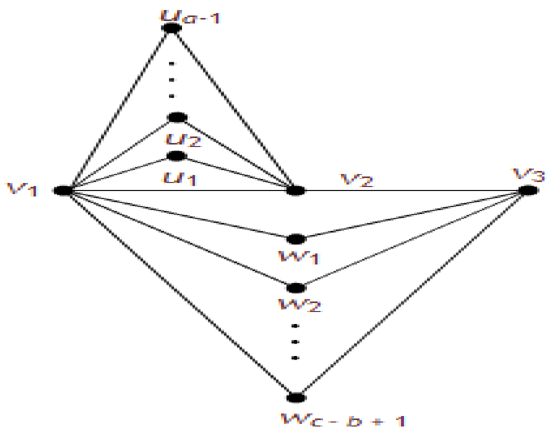

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Figure 2.5

Case 2. Let $a > 2$ and $b - a = 1$. Since $c > b$, we have $c - b + 1 \geq 2$. Consider the graph G given in Figure 2.5. Then as in Case 1, $M = \{u_1, u_2, \dots, u_{a-1}, u_3\}$ is a minimum edge monophonic set, $M_1 = M \cup \{v_2\}$ is a minimum connected edge monophonic set and $M_2 = V(G) - \{v_1\}$ is a minimal connected edge monophonic set of G so that $m_1(G) = a, m_{1c}(G) = b$ and $m_{1c}^{+}(G) = c$.

Case 3. Let $a = 2$ and $b - a = 1$. Then $b = 3$. Consider the graph G given in Figure 2.6. Then as in Case 1, $M = \{v_1, v_3\}$ is a minimum edge monophonic set, $M_1 = \{v_1, v_2, v_3\}$ is a minimum connected edge monophonic set and $M_2 = V(G) - \{v_1\}$is a minimal connected edge monophonic set of G so that $m_1(G) = a, m_{1c}(G) = b$ and $m_{1c}^{+}(G) = c$

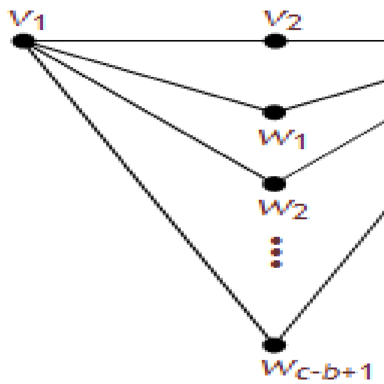

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Figure 2.6

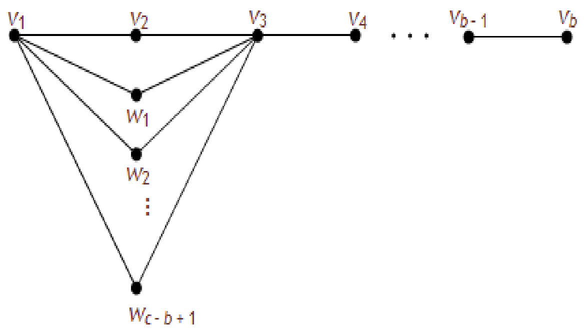

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Figure 2.7

Case 4. Let $a = 2$ and $b - a \geq 2$. Then $b \geq 4$. Consider the graph G given in Figure 2.7. Then as in Case 1, $M = \{v_1, v_b\}$ is a minimum edge monophonic set, $M_1 = \{v_1, v_2, ..., v_b\}$ is a minimum connected edge monophonic set and $M_2 = V(G) - \{v_1\}$ is a minimal connected edge monophonic set of G so that $m_1(G) = a,\ m_{1c}(G) = b$ and $m_{1c}^{+}(G) = c$.

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