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TWO-POINT KRONECKER PRODUCT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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ABSTRACT

In this paper, we establish the existence and uniqueness of the two-point boundary value problem associated with the kronecker product Lyapunov system on time scales

$$(X(t)\otimes Y(t))^{\Delta} = (A(t)\oplus C(t))(X(t)\otimes Y(t)) + (X(\sigma(t))\otimes Y(\sigma(t))(B(t)\oplus D(t)) + (F_1(t)\otimes F_2(t)).$$

$$(M_1\otimes M_2)(X(t_0)\otimes Y(t_0)) + (N_1\otimes N_2)(X(t_1)\otimes Y(t_1)) = \alpha_1\otimes\alpha_2, \quad t_o \le t \le t_1$$

Where A(t), B(t), C(t), D(t), X(t) and Y(t) are square matrices of order n. We assume that the components of A(t), B(t), C(t), D(t) are continuous functions on $[t_0, t_1]$ and F_1, F_2 : $[t_0, t_1]$ $\to$ $R^{n\times n}$ are continuous and $M_1, M_2, N_1, N_2, \alpha_1, \alpha_2$ are constant square matrices of order n. Here modified QR - algorithm and Bartles - Stewart algorithm are used as a tool to compute the constant square matrix ($C_1\otimes C_2$) which is embedded in a peculiar form.

Key words : Kronecker product, Lyapunov system, Boundary value problems, Bartles- Stewart algorithm, QR - algorithm.

AMS (MOS) Subject Classification : 34B05,34B99

1. INTRODUCTION

The kronecker product Lyapunov system arises in a number of areas of Applied Mathematics such as dynamical programming, optimal filters, filter design, design of receding horizon control strategies for multivariate systems and system engineering. The importance of kronecker product Lyapunov boundary value problems has gained momentum due to its economical computations rather than sparse matrices which are usually uneconomical in computational aspects. The study of time scales gained momentum in various disciplines due to its unified approach of studying both continuous and discrete differential systems simultaneously. In this paper, we are concerned about existence and uniqueness of solutions to two point boundary value problem associated with kronecker product Lyapunov systems. In 1991, Murty K.N., and Prasad K.R. [2] constructed the solution of the non-homogeneous Lyapunov system by variation of parameters formula. In 1992, Murty K.N., Howel, G.W. and Sivasundaram. S [3] established existence and uniqueness of solutions to two and multipoint boundary value problems associated with non-linear Lyapunov system. In 2001, Murty, K.N., Lakshmi, V.N. and Ajita, S [5] obtained the existence and uniqueness of solutions of the kronecker product initial value problem involving method of least squares.

In this paper, we consider the non-homogenous kronecker product Lyapunov system of differential equation on time scales:

$$(X(t)\otimes Y(t))^{\Delta} = (A(t)\oplus C(t))(X(t)\otimes Y(t)) + (X(\sigma(t))\otimes Y(\sigma(t)))(B(t)\oplus D(t)) + (F_1(t)\otimes F_2(t)) \quad (1.1)$$

Where A(t), B(t), C(t), D(t), X(t) and Y(t) are square matrices of order n. We assume that the components of A(t), B(t), C(t), D(t) are continuous functions on $[t_0, t_1]$ and F_1, F_2 : $[t_0, t_1] \to R^{n\times n}$ are continuous. The Kronecker product and kronecker sum are denoted by ⊗ and ⊕ respectively. This paper is organized as follows : In section 2, we develop the general solution of the homogeneous kronecker product Lyapunov system corresponding to (1.1) in terms of fundamental matrix solutions. We also obtain a particular solution of non-homogeneous kronecker product Lyapunov system (1.1) by variation of parameters formula. Section 3, deals with the kronecker product two-point boundary value problem associated with (1.1) satisfying the boundary condition

$$(M_1\otimes M_2)(X(t_0)\otimes Y(t_0)) + (N_1\otimes N_2)(X(t_1)\otimes Y(t_1)) = \alpha_1\otimes\alpha_2 \quad (1.2)$$

Where $M_1, M_2, N_1, N_2, \alpha_1, \alpha_2$ are constant square matrices of order n. Here modified QR - algorithm and Bartles - Stewart algorithm are used as a tool to compute the constant square matrix $(C_1\otimes C_2)$ which is embedded in a peculiar form.

2. Generalsolution Of The Kronecker Product Lyapunov System

In this section, we give general form of the solution of the homogeneous kronecker product Lyapunov system on time scales

$$(X(t)\otimes Y(t))^{\Delta} = (A(t)\oplus C(t))(X(t)\otimes Y(t)) + (X(\sigma(t))\otimes Y(\sigma(t)))(B(t)\oplus D(t)) \quad (2.1)$$

in terms of fundamental matrix solutions of

$$(X(t)\otimes Y(t))^{\Delta} = (A(t)\oplus C(t))(X(t)\otimes Y(t)) \quad (2.2)$$

$$(X(t)\otimes Y(t))^{\Delta} = (B(t)\oplus D(t))^{\Delta}(X(\sigma(t))\otimes Y(\sigma(t))) \quad (2.3)$$

and thereby obtain the general solution of the non-homogeneous kronecker product Lyapunov system (1.1) in terms of fundamental matrix solutions by using the method of variation of parameters.

Theorem 2.1 : $(Y_1(t)\otimes Z_1(t))$ is a fundamental matrix solution for the kronecker product system (2.2) if and only if $Y_1(t)$ and $Z_1(t)$ are fundamental matrix solutions for the systems $X^{\Delta}(t)=A(t)X(t)$ and $Y^{\Delta}(t)=C(t)Y(t)$ respectively.

Corollary 2.1 : $(Y_2(t)\otimes Z_2(t))$ is a fundamental matrix solution of (2.3) if and only if $Y_2(t)$ and $Z_2(t)$ are fundamental matrix solutions for the systems $X^{\Delta}(t)=B^*(t)X(\sigma(t))$ and $Y^{\Delta}(t)=D^*(t)Y(\sigma(t))$ respectively.

Theorem 2.2 : Any solution of the homogeneous kronecker product Lyapunov system (2.1) is of the form

where C_1, C_2 are constant square matrices of order n. $Y_1(t), Z_1(t), Y_2(t)$ and $Z_2(t)$ are fundamental matrix solutions of $X^{\Delta}(t)=A(t)X(t), Y^{\Delta}(t)=C(t)Y(t)$, $X^{\Delta}(t)=B^*(t)X(\sigma(t))$ and $Y^{\Delta}(t)=D^*(t)Y(\sigma(t))$ respectively.

Proof : We seek a solution of the homogeneous kronecker product Lyapunov system (2.1) in the form

$$(X(t)\otimes Y(t))=(Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t)),$$

where $K_1(t), K_2(t)$ are square matrices of order n. Then

$$(Y_1(t)\otimes Z_1(t))^{\Delta}(K_1(t)\otimes K_2(t))+(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(t)\otimes K_2(t))^{\Delta}$$

$$=(A(t)\oplus C(t))(Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t))+(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(\sigma(t))\otimes K_2(\sigma(t)))(B(t)\oplus D(t))$$

i.e., $(K_1(t)\otimes K_2(t))^{\Delta}=(K_1(\sigma(t))\otimes K_2(\sigma(t)))(B(t)\oplus D(t))$

i.e.,

$$(K_1(t)\otimes K_2(t))^{\Delta^*}=(B(t)\oplus D(t))^*(K_1(\sigma(t))\otimes K_2(\sigma(t)))^*$$

Since $(Y_2(t)\otimes Z_2(t))$ is a fundamental matrix solution of (2.3), it follows that there exists a constant square matrix $(C_3\otimes C_4)$ such that

$$(K_1(t)\otimes K_2(t))^*=(Y_2(t)\otimes Z_2(t))(C_3\otimes C_4)$$

$$\Leftrightarrow (K_1(t)\otimes K_2(t))=(C_3^*\otimes C_4^*)(Y_2^*(t)\otimes Z_2^*)$$

Hence,

$$X(t)\otimes Y(t)=(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^*(t)\otimes Z_2^*(t))$$

(Take $C_3^*=C_1, C_4^*=C_2$)

Theorem2.3:

$(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^*(t)\otimes Z_2^*(t))$ is a solution of the Lyapunov system (2.1) if and only if $Y_1(t)C_1Y_2^*(t)$ is a solution of

$$X^{\Delta}(t)=A(t)X(t)+X(\sigma(t))B(t) \quad (2.4)$$

and $Z_1(t)C_2Z_2^*(t)$ is a solution of

$$Y^{\Delta}=C(t)Y(t)+Y(\sigma(t))D(t) \quad (2.5)$$

Where $Y_1(t), Z_1(t), Y_2(t)$ and $Z_2(t)$ are fundamental matrix solutions of $X^{\Delta}(t)=A(t)X(t), Y^{\Delta}(t)=C(t)Y(t)$, $X^{\Delta}(t)=B^*(t)X(\sigma(t))$ and $Y^{\Delta}(t)=D^*(t)Y(\sigma(t))$

Theorem 2.4 : Any solution of the kronecker product Lyapunov system (1.1) is of the form

$$X(t)\otimes Y(t)=(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^*(t)\otimes Z_2^*(t))+(\bar{X}(t)\otimes\bar{Y}(t))$$

where $\bar{X}(t)\otimes\bar{Y}(t)$ is a particular solution of (1.1).

Proof : It can easily be verified that $X(t)\otimes Y(t)$ definedby

$$X(t)\otimes Y(t)=(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^*(t)\otimes Z_2^*(t))+\bar{X}(t)\otimes\bar{Y}(t)$$

is a solution of (1.1).

Now to prove that every solution of (1.1) is of this form, let $X(t)\otimes Y(t)$ be any solution of (1.1) and $\overline{X}(t)\otimes\overline{Y}(t)$ be a particular solution of (1.1). Then $(X(t)\otimes Y(t))-(\overline{X}(t)\otimes\overline{Y}(t))$ is a solution of the homogeneous kronecker product Lyapunov system (2.1). Any solution of the homogeneous kronecker product Lyapunov system (2.1) is given by $(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^*(t)\otimes Z_2^*(t))$, where $(C_1\otimes C_2)$ is constant square matrix of order $(n^2\times n^2)$. Thus

$$(X(t)\otimes Y(t))-(\overline{X}(t)\otimes\overline{Y}(t))=$$
$$(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^*(t)\otimes Z_2^*(t))$$
$$\Leftrightarrow X(t)\otimes Y(t)=$$
$$(Y_1(t)\otimes Z_1(t))(C_1\otimes C_2)(Y_2^*(t)\otimes Z_2^*(t))+$$
$$\overline{X}(t)\otimes\overline{Y}(t).$$

Hence the proof of the theorem is complete.

Theorem 2.5 : A particular solution of the kronecker product Lyapunov system (1.1) is given by

$$\overline{X}(t)\otimes\overline{Y}(t)=(Y_1(t)\otimes Z_1(t))$$
$$\left[\int_{t_0}^{t}(Y_1(s)\otimes Z_1(s))^{-1}(F_1(s)\otimes F_2(s))(Y_2^*(s)\otimes Z_2^*(s))^{-1}ds\right]$$
$$(Y_2^*(t)\otimes Z_2^*(t))$$

Proof : We seek a particular solution of the kronecker product Lyapunov system (2.1) in the form

$$\overline{X}(t)\otimes\overline{Y}(t)=(Y_1(t)\otimes Z_1(t))$$
$(K_1(t)\otimes K_2(t))(Y_2^*(t)\otimes Z_2^*(t))$, where $K_1(t), K_2(t)$ are square matrices of order n. Then substituting $\overline{X}(t)\otimes\overline{Y}(t)$ in (1.1) we get,

$$((Y_1(t)\otimes Z_1(t))\,(K_1(t)\otimes K_2(t))(Y_2^*(t)\otimes Z_2^*(t)))^{\Delta}$$
$$=(A(t)\oplus C(t))(Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t))$$
$$(Y_2^*(t)\otimes Z_2^*(t))\quad +$$
$$(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(\sigma(t))\otimes K_2(\sigma(t)))$$
$$(Y_2^*(\sigma(t))\otimes Z_2^*(\sigma(t)))$$
$$(B(t)\oplus D(t))+(F_1(t)\otimes F_2(t))$$

$$(Y_1(t)\otimes Z_1(t))^{\Delta}(K_1(t)\otimes K_2(t))(Y_2^*(t)\otimes Z_2^*(t))$$
$$+(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(t)\otimes K_2(t))^{\Delta}(Y_2^*(t)\otimes Z_2^*(t))$$
$$(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(\sigma(t))\otimes K_2(\sigma(t)))(Y_2^*(t)\otimes Z_2^*(t))^{\Delta}$$
$$=(A(t)\oplus C(t))(Y_1(t)\otimes Z_1(t))(K_1(t)\otimes K_2(t))$$
$$(Y_2^*(t)\otimes Z_2^*(t))+$$
$$(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(\sigma(t))\otimes K_2(\sigma(t)))$$
$$(Y_2^*(\sigma(t))\otimes Z_2^*(\sigma(t)))$$
$$(B(t)\oplus D(t))+(F_1(t)\otimes F_2(t))$$

$$\Leftrightarrow\ \begin{array}{l}(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))(K_1(t)\otimes K_2(t))'\\(Y_2^*(t)\otimes Z_2^*(t))=(F_1(t)\otimes F_2(t))\end{array}$$

$$\Leftrightarrow\ \begin{array}{l}(K_1(t)\otimes K_2(t))^{\Delta}=(Y_1(\sigma(t))\otimes Z_1(\sigma(t)))^{-1}\\(F_1(t)\otimes F_2(t))(Y_2^*(t)\otimes Z_2^*(t))^{-1}\end{array}$$

$$\Leftrightarrow K_1(t)\otimes K_2(t)=$$

$$\left[\begin{array}{l}\int_{t_0}^{t}(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}(F_1(s)\otimes F_2(s))\\(Y_2^*(s)\otimes Z_2^*(s))^{-1}\Delta s\end{array}\right]$$

.

Hence, $\overline{X}(t)\otimes\overline{Y}(t)=$

$$\left(Y_1(t)\otimes Z_1(t)\right)\left[\int_{t_0}^{t}\left(Y_1(\sigma(s))(s)\otimes Z_1(\sigma(s))\right)^{-1}\left(F_1(s)\otimes F_2(s)\right)\left(Y_2^*(s)\otimes Z_2^*(s)\right)^{-1}\Delta s\right]\left(Y_2^*(t)\otimes Z_2^*(t)\right)$$

Theorem 2.6 : If a particular solution of $X'(t)=A(t)X(t)+X(\sigma(t))B(t)+F_1(t)$ is given by

$$\overline{X}(t)=Y_1(t)\left[\int_{t_0}^{t}Y_1^{-1}(\sigma(s))F_1(s)Y_2^{*^{-1}}(s)\Delta s\right]Y_2^*(t)$$

and a particular solution of $Y'(t)=C(t)Y(t)+Y(\sigma(t))D(t)+F_2(t)$ is given by

$$\overline{Y}(t)=Z_1(t)\int_{t_0}^{t}Z_1^{-1}(\sigma(s))F_2(s)\,Z_2^{*^{-1}}(s)\Delta s Z_2^*(t),$$

then a particular solution of the kronecker product Lyapunov system (1.1) is given by $\overline{X}(t)\otimes\overline{Y}(t)=$

$$\left(Y_1(t)\otimes Z_1(t)\right)\left[\int_{t_0}^{t}\left(Y_1(\sigma(s))(s)\otimes Z_1(\sigma(s))\right)^{-1}\left(F_1(s)\otimes F_2(s)\right)\left(Y_2^*(s)\otimes Z_2^*(s)\right)^{-1}\Delta s\right]\left(Y_2^*(t)\otimes Z_2^*(t)\right)$$

The general solution of the non-homogeneous kronecker product Lyapunov system (1.1) is given by

$X(t)\otimes Y(t)=$

$$\left(Y_1(t)\otimes Z_1(t)\right)\left(C_1\otimes C_2\right)\left(Y_2^*(t)\otimes Z_2^*(t)\right)+\left(Y_1(t)\otimes Z_1(t)\right)\left[\int_{t_0}^{t}\left(Y_1(\sigma(s))(s)\otimes Z_1(\sigma(s))\right)^{-1}\left(F_1(s)\otimes F_2(s)\right)\left(Y_2^*(s)\otimes Z_2^*(s)\right)^{-1}\Delta s\right]\left(Y_2^*(t)\otimes Z_2^*(t)\right)$$

3. Two - Point Boundary Value Problem

In this section, we establish the existence and uniqueness of solution for two point boundary value problem associated with kronecker product Lyapunov system (1.1). Here we obtain the solution of two point boundary value problem associated with (1.1) satisfying the boundary condition matrix (1.2), using modified QR - algorithm and Bartles - Stewart algorithm.

The general solution of kronecker product Lyapunov system (1.1) is given by

$X(t)\otimes Y(t)=$

$$\left(Y_1(t)\otimes Z_1(t)\right)\left(C_1\otimes C_2\right)\left(Y_2^*(t)\otimes Z_2^*(t)\right)+\left(Y_1(t)\otimes Z_1(t)\right)\left[\int_{t_0}^{t}\left(Y_1(\sigma(s))(s)\otimes Z_1(\sigma(s))\right)^{-1}\left(F_1(s)\otimes F_2(s)\right)\left(Y_2^*(s)\otimes Z_2^*(s)\right)^{-1}\Delta s\right]\left(Y_2^*(t)\otimes Z_2^*(t)\right)$$

Substituting the general solution in the boundary condition matrix (1.2) we get,

$$\left(M_1\otimes M_2\right)\left(Y_1(t_0)\otimes Z_1(t_0)\right)\left(C_1\otimes C_2\right)\left(Y_2^*(t_0)\otimes Z_2^*(t_0)\right)+\left(N_1\otimes N_2\right)\left(Y_1(t_1)\otimes Z_1(t_1)\right)\left(C_1\otimes C_2\right)\left(Y_2^*(t_1)\otimes Z_2^*(t_1)\right)$$

$$+(N_1 \otimes N_2)(Y_1(t_1) \otimes Z_1(t_1))$$
$$\left[\int_{t_0}^{t_1} (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} (F_1(s) \otimes F_2(s)) (Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s\right] (Y_2^*(t_1) \otimes Z_2^*(t_1)) = \alpha_1 \otimes \alpha_2 .$$

(3.1)

The above equation (3.1) is equivalent to

$$(A_1 \otimes A_2)(C_1 \otimes C_2)(B_1 \otimes B_2) + (P_1 \otimes P_2)(C_1 \otimes C_2)(Q_1 \otimes Q_2) = \omega \qquad (3.2)$$

where

$$A_1 = M_1 Y_1(t_0), A_2 = M_2 Z_1(t_0), B_1 = Y_2^*(t_0), B_2 = Z_2^*(t_0), P_1 = N_1 Y_1(t_1)$$

$P_2 = N_2 Z_1(t_1), Q_1 = Y_2^*(t_1), Q_2 = Z_2^*(t_1)$ and

$$\omega = (\alpha_1 \otimes \alpha_2) - (N_1 \otimes N_2)(Y_1(t_1) \otimes Z_1(t_1)) \int_{t_0}^{t_1} (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} (F_1(s) \otimes F_2(s)) . (Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s \} (Y_2^*(t_1) \otimes Z_2^*(t_1))$$

A_1, A_2, B_1, B_2, P_1, P_2, Q_1 and Q_2 are square matrices of order n and ω is a square matrix of order n^2. Note that $(B_1 \otimes B_2)$ and $(Q_1 \otimes Q_2)$ are non-singular matrices and we shall be concerned with the general form of $C_1 \otimes C_2$ satisfying the condition (3.2). Using kronecker product of matrices, (3.2) can be written as a system of vector equations

$$G\,(\overline{C_1 \otimes C_2}) = \overline{\omega} \qquad (3.3)$$

where G = $\left[(A_1 \otimes A_2) \otimes (B_1 \otimes B_2)^* + (P_1 \otimes P_2) \otimes (Q_1 \otimes Q_2)^*\right]$ is $(n^4 x n^4)$ matrix, $(\overline{C_1 \otimes C_2})$ and $\overline{\omega}$ are n^4 column vectors corresponding to the matrices $C_1 \otimes C_2$ and ω. In fact, by viewing (3.2) as a system of n^4 - scalar equations for the elements of $C_1 \otimes C_2$, (3.3) is exactly the same set of equations written in vector system. In order to make pronouncements about existence and uniqueness techniques for the solution of (3.3), we need some information about the eigenvalues of G. We denote the set of all eigenvalues of the matrix A as σ (A), the spectrum of A.

Case(i) If $(A_1 \otimes A_2)$ and $(B_1 \otimes B_2)$ are non singular, then (3.2) is equivalent to

$$C_1 \otimes C_2 - (R_1 \otimes R_2)(C_1 \otimes C_2)(S_1 \otimes S_2) = L \qquad (3.4)$$

where

$R_1 = A_1^{-1}P_1, R_2 = A_2^{-1}P_2, S_1 = Q_1B_1^{-1}, S_2 = Q_2B_2^{-1}$,

L = $(A_1 \otimes A_2)^{-1}$ ω $(B_1 \otimes B_2)^{-1}$.

Using the result on kronecker product of matrices (3.4) can be written as

$$C_1 \otimes C_2 - (R_1 \otimes R_2)(C_1 \otimes C_2)(S_1 \otimes S_2) = L$$
$$\Leftrightarrow \left[(I \otimes I) - (R_1 \otimes R_2) \otimes (S_1 \otimes S_2)^*\right](\overline{C_1 \otimes C_2}) = \overline{L}$$
$$\Leftrightarrow I(\overline{C_1 \otimes C_2}) - (R_1 \otimes R_2) \otimes (S_1 \otimes S_2)^* (\overline{C_1 \otimes C_2}) = \overline{L}$$
$$\Leftrightarrow (\overline{C_1 \otimes C_2}) - (G_1 \otimes G_2)(\overline{C_1 \otimes C_2}) = \overline{L}$$

where $G_1 = R_1 \otimes R_2$, $G_2 = S_1 \otimes S_2$.

Now,putting $(C_1 \otimes C_2) = L + (R_1 \otimes R_2)(C_1 \otimes C_2)(S_1 \otimes S_2)$ in the second term on LHS of (3.4), we get the following equivalent statements

$$(C_1 \otimes C_2) - (R_1 \otimes R_2)\left[L + (R_1 \otimes R_2)(C_1 \otimes C_2)(S_1 \otimes S_2)\right](S_1 \otimes S_2) = L$$

$$\Leftrightarrow (\overline{C_1 \otimes C_2}) - (G_1 \otimes G_2)\left[\overline{L} + (G_1 \otimes G_2)(\overline{C_1 \otimes C_2})\right] = \overline{L}$$

$$C_1 \otimes C_2 - (R_1 \otimes R_2)^2 (C_1 \otimes C_2)(S_1 \otimes S_2)^2$$
$$= L + (R_1 \otimes R_2) L (S_1 \otimes S_2)$$
$$\Leftrightarrow (\overline{C_1 \otimes C}_2) - (G_1 \otimes G_2)^2 (\overline{C_1 \otimes C}_2)$$
$$= \overline{L} + (G_1 \otimes G_2)\overline{L}$$

$$C_1 \otimes C_2 - (R_1 \otimes R_2)^m (C_1 \otimes C_2)(S_1 \otimes S_2)^m \quad \ldots\ldots$$
$$= L + (R_1 \otimes R_2) L (S_1 \otimes S_2) + \ldots\ldots$$
$$+ (R_1 \otimes R_2)^{m-1} L (S_1 \otimes S_2)^{m-1}$$

$$\Leftrightarrow (\overline{C_1 \otimes C}_2) - (G_1 \otimes G_2)^m (\overline{C_1 \otimes C}_2)$$
$$= \overline{L} + (G_1 \otimes G_2)\overline{L} + \ldots\ldots$$
$$\ldots. + (G_1 \otimes G_2)^{m-1} \overline{L}.$$

If the spectral radii of $(R_1 \otimes R_2)$ and $(S_1 \otimes S_2)$ are such that $\rho(R_1 \otimes R_2)\ \rho(S_1 \otimes S_2) < 1$, then $(R_1 \otimes R_2)^m (C_1 \otimes C_2)(S_1 \otimes S_2)^m \to 0$ as $m \to \infty$. In this case,

$$(C_1 \otimes C_2) = L + \sum_{m=1}^{\infty} (R_1 \otimes R_2)^m L (S_1 \otimes S_2)^m$$
$$= (A_1 \otimes A_2)^{-1} \omega (B_1 \otimes B_2)^{-1} +$$
$$\sum_{m=1}^{\infty} (R_1 \otimes R_2)^m (A_1 \otimes A_2)^{-1} \omega (B_1 \otimes B_2)^{-1} (S_1 \otimes S_2)^m.$$

Hence the solution $(X(t) \otimes Y(t))$ of (1.1) is of the form

$$(X(t) \otimes Y(t)) = (Y_1(t) \otimes Z_1(t)) [M_1 Y_1(t_0) \otimes M_2 Z_1(t_0)]^{-1}$$
$$\Big\{ \alpha_1 \otimes \alpha_2 - (N_1 \otimes N_2)(Y_1(t_1) \otimes Z_1(t_1))$$
$$\left[\int_{t_0}^{t_1} (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} (F_1(s) \otimes F_2(s)) (Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s \right]$$
$$(Y_2^*(t_1) \otimes Z_2^*(t_1)) \Big\}$$
$$(Y_2^*(t_0) \otimes Z_2^*(t_0))^{-1} (Y_2^*(t) \otimes Z_2^*(t)) +$$
$$(Y_1(t) \otimes Z_1(t))$$
$$\sum_{m=1}^{\infty} (-1)^m \Big[(M_1 Y_1(t_0))^{-1} (N_1 Y_1(t_1)) \otimes$$
$$(M_2 Z_1(t_0))^{-1} (N_2 Z_1(t_1)) \Big]^m$$
$$[M_1 Y_1(t_0) \otimes M_2 Z_1(t_0)]^{-1}$$
$$\Big\{ \alpha_1 \otimes \alpha_2 - (N_1 \otimes N_2)(Y_1(t_1) \otimes Z_1(t_1))$$
$$\left[\int_{t_0}^{t} (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} (F_1(s) \otimes F_2(s)) \right.$$
$$\left. (Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s \right] (Y_2^*(t_1) \otimes Z_2^*(t_1)) \Big\}$$
$$(Y_2^*(t_0) \otimes Z_2^*(t_0))^{-1}$$

$$\left[Y_2^*(t_1) Y_2^{*^{-1}}(t_0) \otimes Z_2^*(t_1) Z_2^{*^{-1}}(t_0) \right]^m (Y_2^*(t) \otimes Z_2^*(t))$$
$$+ (Y_1(t) \otimes Z_1(t))$$
$$\left[\int_{t_0}^{t} (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} (F_1(s) \otimes F_2(s)) \right.$$
$$\left. (Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s \right]$$
$$(Y_2^*(t) \otimes Z_2^*(t))$$

Case (ii) Suppose $(A_1 \otimes A_2)$ is invertible. Then the system of equations (3.2) is equivalent to

$$(P \otimes Q)(C_1 \otimes C_2) + (C_1 \otimes C_2)(R \otimes S) = \eta \quad (3.5)$$

where $P = A_1^{-1} P_1, Q = A_2^{-1} P_2, R = B_1 Q_1^{-1}, S = B_2 Q_2^{-1}$ and $\eta = (A_1 \otimes A_2)^{-1} \omega (Q_1 \otimes Q_2)^{-1}$ one of the most effective methods of solving the matrix equation (3.5) is the Bartle - Stewart algorithm. Key to this technique is the orthogonal reduction of $(P \otimes Q)$ and $(R \otimes S)$ into triangular form using

QR - algorithm for eigenvalues. The method of finding the general solution to the system (3.5), we follow the following method :

Let $P\otimes Q$, $R\otimes S \in R^{n^2\times n^2}$ be given matrices and define the linear transformation $\psi : R^{n^2\times n^2} \to R^{n^2\times n^2}$ by

$\psi(C_1\otimes C_2)=$

$$(P\otimes Q)(C_1\otimes C_2)+(C_1\otimes C_2)(R\otimes S) = \eta$$

(3.6) This linear transformation is non-singular; if and only if $(P\otimes Q)$, $(R\otimes S)$ have no common eigenvalues. If λ, μ are two eigenvalues of P and Q respectively, ie, $\lambda \in \sigma(p)$, $\mu \in \sigma(Q)$ with corresponding eigen vectors u and v of P and Q respectively, if ν and ρ are two eigen values of R and S, ie., $\nu \in \sigma(R)$, $\rho \in \sigma(s)$ with corresponding eigen vectors $\bar{u}$ and $\bar{v}$ respectively, then $\lambda_i \mu_j \in \sigma(P\otimes Q)$, $(u\otimes v) \in C^{n\times n}$ is a corresponding eigen vector of $(P\otimes Q)$ and $\nu_i \rho_j \in \sigma(R\otimes S)$, $(\bar{u}\otimes\bar{v}) \in C^{n\times n}$ is a corresponding eigen vector of $(R\otimes S)$. Every eigen value of $(P\otimes Q)$ and $(R\otimes S)$ arises as such as a product of eigen values of $(P\otimes Q)$ and $(R\otimes S)$.

If $\sigma(P) = \{\lambda_1,\lambda_2\ldots\lambda_n\}$,

$\sigma(Q) = \{\mu_1,\mu_2\ldots\ldots\mu_n\}$ then

$\sigma(P\otimes Q)=\{\lambda_1\mu_j : i,j = 1,2\ldots\ldots n\}$, and

$\sigma(s) = \{\rho_1,\rho_2\ldots\ldots\rho_n\}$ then

$\sigma(R\otimes S) = \{\nu_i\rho_j : i,j = 1,2\ldots\ldots n\}$.

Now

$$(P\otimes Q)(u\otimes v)(\bar{u}\otimes\bar{v})^T$$
$$=\lambda\mu(u\otimes v)(\bar{u}\otimes\bar{v})^T \text{ and}$$
$$(u\otimes v)(\bar{u}\otimes\bar{v})^T(R\otimes S) = \nu\rho$$
$$(u\otimes v)(\bar{u}\otimes\bar{v})^T.$$

Hence,

$$(P\otimes Q)(u\otimes v)(\bar{u}\otimes\bar{v})^T + (u\otimes v)(\bar{u}\otimes\bar{v})^T(R\otimes S)$$
$$=(\lambda\mu+\nu\rho)(u\otimes v)(\bar{u}\otimes\bar{v})^T.$$

This implies $(\lambda\mu+\nu\rho)$ is an eigen value of the system (3.6), which can therefore be solved if and only if $(\lambda_i\mu_j+\nu_i\rho_j)\neq 0$ for all i, j = 1,2, ... n.

Now reducing $(P\otimes Q)$ and $(R\otimes S)$ to the diagonal form by using similarity transformations, we get

$$(U_1^{-1}\otimes U_2^{-1})(P\otimes Q)(U_1\otimes U_2) =$$
Diag
$$\{\lambda_1\mu_1\ldots\lambda_1\mu_n,\lambda_2\mu_1\ldots\lambda_2\mu_n\ldots\lambda_n\mu_1\ldots\lambda_n\mu_n\} = D_1\otimes D_2$$
$$(U_1^{-1}\otimes U_2^{-1})(R\otimes S)(V_1\otimes V_2) =$$
Diag
$$\{\nu_1\rho_1\ldots,\nu_1\rho_n\ldots\nu_2\rho_1\ldots\nu_2\rho_n\ldots\nu_n\rho_1\ldots\nu_n\rho_n\} = D_3\otimes D_4$$

Then equation (3.6) is equivalent to

$$(U_1^{-1}\otimes U_2^{-1})(R\otimes S)(U_1\otimes U_2)(U_1^{-1}\otimes U_2^{-1})$$
$$(C_1\otimes C_2)(V_1\otimes V_2)+$$
$$(U_1^{-1}\otimes U_2^{-1})(C_1\otimes C_2)(V_1\otimes V_2)(V_1^{-1}\otimes V_2^{-1})$$
$$(R\otimes S)(V_1\otimes V_2)$$
$$=(U_1^{-1}\otimes U_2^{-1})\eta(V_1\otimes V_2)$$

Now to solve the system, we proceed as follows :

Step 1 : By using similarity transformations reduce $(P\otimes Q)$, $(R\otimes S)$ to diagonal form

$$D_1\otimes D_2 = (U_1^{-1}\otimes U_2^{-1})(P\otimes Q)(U_1\otimes U_2)$$
$$and D_3\otimes D_4 = (V_1^{-1}\otimes V_2^{-1})(R\otimes S)(V_1\otimes V_2)$$

Step2: Solve

$(U_1\otimes U_2)\ E = \omega\ (V_1\otimes V_2)$ for E.

Step 3 : Solve the transformed system

$$(D_1\otimes D_2)(X_1\otimes Y_1)+(X_1\otimes Y_1)(D_3\otimes D_4)$$
$$=E \ \ for\ (X_1\otimes Y_1)$$

Step 4 : Solve the system

$$(C_1\otimes C_2)(V_1\otimes V_2)=(U_1\otimes U_2)(X_1\otimes Y_1)$$
$$for\ (C_1\otimes C_2)$$

From these above steps we get the solution of the system (3.6) and is given by $C_1 \otimes C_2 = (U_1 \otimes U_2)(X_1 \otimes Y_1)(V_1^{-1} \otimes V_2^{-1})$,

where $\left(X_{ij} \otimes Y_{ij}\right) = \dfrac{e_{ij}}{\lambda_i \mu_j + \nu_i \rho_j}$,

$E = e_{ij} = \left(U_1^{-1} \otimes U_2^{-1}\right)\eta\left(V_1 \otimes V_2\right)$.

Now by substituting the general form of $C_1 \otimes C_2$ in the general solution $(X(t) \otimes Y(t))$ of (1.1),

we have

$$\begin{aligned}
&(X(t) \otimes Y(t)) = (Y_1(t) \otimes Z_1(t))(U_1 \otimes U_2) \\
&(X_1 \otimes Y_1)(V_1^{-1} \otimes V_2^{-1})(Y_2^*(t) \otimes Z_2^*(t)) \\
&+ (Y_1\ (t) \otimes Z_1(t)) \\
&\int_{t_0}^{t} (Y_1\ (\sigma(s)) \otimes Z_1\ (\sigma(s)))^{-1} (F_1\ (s) \otimes F_2\ (s)). \\
&(Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s (Y_2^*(t) \otimes Z_2^*(t))
\end{aligned}$$

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