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GENERALIZED STIELTJES TRANSFORM OF VECTOR VALUED FUNCTIONS

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ABSTRACT:

In this paper we introduced Generalized Stieltjes Transform for vector valued functions and inversion theorem for generalized Stieltjes transform is proved.

Keywords: Stieltjes Transform, Banach space.

1. INTRODUCTION:

The method of function transformations has been used successfully in solving many problems in engineering, mathematics, physics and applied mathematics. Function transformations include, but are not limited to the well known technique of linear function transformations .A function transformation simply means a mathematical operation through which a real or complex –valued function f is transformed into another function F ,or into a sequences of numbers ,or more generally into a set of data .This is done by taking Integral transform.

In recent years the theory of vector valued transform has become an important tool in connection with study of evolutionary problems [1, 2, 5, 11].

Motivated by the work C. Lizama and H.Prado [7] we studied generalized Stieltjes transform for vector valued function.

Various generalizations of the Stieltjes transform

$$F(s) = \int_0^\infty f(t)(s+t)^{-1} dt \qquad (1.1)$$

have been studied by many mathematicians from time to time. One of such is given by

$$F(s) = S^{\rho-1}\Gamma_\rho \int_0^\infty f(t)(s+t)^{-\rho} dt$$

or $$F(s) = \int_0^\infty f(t)(s+t)^{-\rho} dt$$

for $(\rho > 0)$

In this paper we have extended the inversion theorem for vector valued generalized Stieltjes transform. The purpose of the present paper to introduce inversion theorem of generalized Stieltjes transform of vector valued function which Stieltjes transform of vector valued function [7]will be come out as particular case ,and thus to extend its use to widerclass.

2. Definition : Let X denotes a Banach space, and || || is the norm. The space $L^p((0,\infty);X)$ $0 \le p \le \infty$, consists of all the functions $f:(0,\infty) \to X$ which are measurable and

$\|f\|_p = (\int_0^\infty \|f(s)\|^p ds)^{1/p} < \infty \qquad 1 \le p < \infty.$

If $p = \infty$ we define $\|f\|_\infty$=ess.sup$\|f(s)\|$.

The p-variation of a Bochner measurable function $f:[0,\infty) \to X$ is defined by

$$\|f\|_{V_p([0\infty);X)} = \begin{cases} \sup\left\{\left(\sum_{k=1}^{n} \frac{\|f(t_k)-f(t_{k-1})\|^p}{|t_k-t_{k-1}|^{p-1}}\right)^{\frac{1}{p}} : 0 \le t_0 < t_1 < \cdots < t_n < \infty\right\} & for\ 1 \le p < \infty \\ \sup\left\{\frac{\|f(t)-f(s)\|}{|t-s|} : \quad 0 \le t < s < \infty\right\} & if\ p = \infty. \end{cases}$$

We define $V_p([0,\infty);X)$, $1 \le p < \infty$, as the space of the function f with finite p-variation and f(0)=0 ([10]). If x=$\mathbb{C}$ we write L^p and V_p simply.

We know that a Banach space X satisfy the Radon-Nikodym property if the fundamental theorem of calculus holds for absolutely continuous function f with values in X, c.f. [1].

A characterization of the Radon-Nikodym property states that if the space X has the Radon-Nikodym property then for p> 1 $L^p((0,\infty);X)$ and $V_p([0,\infty);X)$ are isometrically isomorphic, the isomorphism, given by the antiderivative I(f)(t)=$\int_0^t f(s)ds$. If X is an arbitrary Banach space, the antiderivative defines an isometric embedding of $L_p((0,\infty);X)$, p > 1, onto a subspace (see[3]).

Theorem 2.1 (Riesz-Stieltjes Representation)

Let X be a Banach space and let $S: L^q \to X$ be a bounded linear operator. Then there exists a unique $F \in V_p([0,\infty);X)$ such that

$$S(g) = \int_0^\infty g(s) d\,F(s)$$

for every $g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$

The proof is same the case X=$\mathbb{C}$.

Corollary 2.1 For p > 1, $q^{-1} - (1 - p^{-1})$, there is an isometric isomorphism between $V_p([0,\infty);X)$ and B(L^q,X).

3. The vector-valued generalized Stieltjes transform

Let X be a Banach space, and let $u:[0,\infty) \to X$ be a Bochner measurable function. Then we define the Stieltjes transform of u by

$$S_\rho(u)(\lambda) = \int_0^\infty \frac{u(s)}{(\lambda+s)^\rho} ds \qquad \ldots(3.1)$$

whenever the latter integral exists. If the integral (3.1) is convergent for some $\lambda > 0$ and u is locally integrable function on $[0, \infty)$ then

$\int_0^t \frac{u(s)}{(\lambda+s)^\rho} ds$ is convergent for every $\lambda > 0$. Furthermore, if the integral (3.1) converges for some complex number $\lambda_0 \in \mathbb{C}_\pi = \mathbb{C}\backslash\{\sigma + i\tau : \sigma \le 0, \tau = 0\}$, then $S_\rho(u)(\lambda)$ exists for all $\lambda \in \mathbb{C}_\pi$.

Proposition 3.1 let u:$[0, \infty) \to X$ be a Bochner integrable function. If there are constants M$> c, 0 < \delta < 1, \left\| \int_0^t u(s)ds \right\| \le Mt^\delta, t > 0$. Then $S_\rho(u)(\lambda)$ exists for every $\lambda \in \mathbb{C}_\pi$.

Proof: Denote $\emptyset(t) = \int_0^t u(s)ds$.

Consider

$$\int_0^t \frac{u(s)}{(\lambda+s)^\rho} ds = \frac{1}{(\lambda+t)^\rho}\emptyset(t) + \int_0^t \frac{\rho}{(\lambda+s)^{\rho+1}}\emptyset(s)ds \qquad \lambda > 0, t > 0$$

Hence

$$\left\| \int_0^t \frac{u(s)}{(\lambda+s)^\rho} ds \right\| \le \frac{Mt^\delta}{(\lambda+t)^\rho} + M\int_0^t \frac{\rho\, s^\delta}{(\lambda+s)^{\rho+1}} ds$$

$$= \frac{Mt^\delta}{(\lambda+t)^\rho} + \text{M}\left\{\frac{-t^\delta}{(\lambda+t)^\rho} + \delta\int \frac{s^{\delta-1}}{(\lambda+s)^\rho} ds\right\}$$

letting $t \to \infty$ and $0 < \delta < 1$ and $\lambda > 0$. So we obtain the desired assertion.

And we have the following.

Proposition 3.2: Let $u: [0, +\infty) \to X$ be a Bochner integrable function. If the generalized Stieltjes transform for u exists for every $\lambda > 0$, then there is a constant $M > 0$ such that

$$\left\| \int_0^t u(s)ds \right\| \le Mt^\rho, \qquad t \ge 0.$$

Proof : Suppose that $\int_0^t \frac{u(t)}{(\lambda+t)^\rho} dt$ exists, at $\lambda = 1$ and hence all $\lambda > 0$. Then integration by parts yields

$$\int_0^t u(s)ds - \int_0^1 u(s)ds = \int_1^t u(s)ds$$

$$= \int_1^t (s+1)^\rho \frac{u(s)}{(s+1)^\rho} ds$$

$$= (t+1)^\rho \int_1^t \frac{u(s)}{(s+1)^\rho} ds + \int_1^t \frac{\rho\,(\tau+1)^{-\rho-1}}{1}\left[\int_1^\tau \frac{u(s)}{(s+1)^\rho} ds\right] d\tau$$

thus

$$\frac{1}{t^\rho}\int_0^t u(s)ds = \frac{1}{t^\rho}\int_0^1 u(s)ds + \left(1+\frac{1}{t}\right)^\rho \int_1^t \frac{u(s)}{(s+1)^\rho} ds + \frac{1}{t^\rho}\int_1^t \int_1^\rho \frac{\rho\,(\tau+1)^{\rho-1}}{1} \frac{u(s)}{(s+1)^\rho} ds\, d\tau.$$

Now letting $t \to 0$, we obtain

$$\lim_{t\to\infty} \frac{1}{t^\rho}\int_0^t u(s)ds = \left(1+\frac{1}{t}\right)^\rho \int_1^\infty \frac{u(s)}{(s+1)^\rho} ds - \left(1+\frac{1}{t}\right)^\rho \int_1^\infty \frac{u(s)}{(s+1)^\rho} ds = 0$$

Corollary 3.1 If u is generalized Stieltjes transformable, then u is Laplace transformable.

The classical Stieltjes transform can be obtained as the square of the Laplace transform and this property remains true for vector valued generalized Laplace transform.

Theorem 3.1 let u: $[0,\infty) \to X$ be a Bochner integrable

If $F(\lambda) = \lambda^{\rho-1}\Gamma_\rho \int_0^\infty \frac{u(t)}{(\lambda+t)^\rho} dt$ exists for every $\lambda > 0$ then

$$\lambda^{\rho-1}\Gamma_\rho \int_0^\infty \frac{u(t)}{(\lambda+t)^\rho} dt - \int_0^\infty G_{1,2}^{2,0}\left(\lambda t \middle| \begin{matrix} 0 \\ 0, et \end{matrix}\right) \emptyset(t) dt \ , \lambda > 0$$

where

$\emptyset(t) = \int_0^\infty e^{-st} u(s) ds$, $\lambda \geq 0$. See [6]

Lemma 3.1 Let $u \in L^p((0,\infty);X), 0 < \rho < \infty$.

Then $S_\rho(u)(\lambda)$ exists for every $\lambda \in \mathbb{C}_\pi$ and is an infinitely differentiable function.

Proof: Let $q = \frac{p}{p-1}$ and $\lambda \in \mathbb{C}_\pi$. Then by Holder's inequality we obtain

$$\left\| \int_0^\infty \frac{u(s)}{(\lambda+s)^\rho} dt \right\| \leq \left(\int_0^\infty \|u(s)\|^p ds\right)^{\frac{1}{p}} \left(\int_0^\infty \frac{1}{((\lambda+s)^\rho)^q} dt\right)^{\frac{1}{q}}$$

from which the existence of $S_\rho(u)(\lambda)$ follows. This, in turn, implies the uniform convergence of the integral.

$$\lambda^{\rho-1}\Gamma_\rho \int_0^\infty \frac{u(s)}{(\lambda+s)^\rho} ds$$

on any closed bounded region not containing a point of the negative real axis (see [12] Theorem 2b, p.327). Hence it represents an analytic function in the complex cut along the negative real axis.

4. The Inversion of the vector-valued Generalized Stieltjes Transform.

To obtain an inversion formula for the generalized Stieltjes transform we use the Schwartz Space $S_0((0,\infty);X)$ of the rapidly decreasing c^∞ function.

$f: (0,\infty) \to X$ such that $\lim_{t\to\infty} \|t^k D^k f(t)\| = 0$ and $\lim_{t\to 0+} \|t^k D^k f(t)\| = 0$

for every k=0, 1, 2,…

if $p \geq 1$, then

$S_0((0,\infty);X) \subset L^p((0,\infty);X)$. Let $f \in S_0((0,\infty);X)$, then we denote[see 8 P.642]

$$L_k^\rho[f(t)] = \frac{(-1)^{k-1}(2k-1)!\Gamma_\rho}{k!(k-2)!\Gamma_{(2k+\rho-1)}}[t^{2k+\rho-2} f^{(k-1)}(t)]^{(k)}, \ k=2,3,\ldots \qquad \ldots (4.1)$$

here $f \in S_0((0,\infty);X)$ the function $t^{2k+\rho-2} f^{(k-1)}(t)$ also belongs to $S_0((0,\infty);X)$ and $S_0((0,\infty);X)$ is invariant under the operators L_k^ρ. That is S and L_k^ρ commute.

Proposition 4.1: Let $S_0((0,\infty);X)$. Then $L_k^\rho\left(S_\rho(f)\right) = S_\rho\left(L_k^\rho(f)\right)$, $k \geq 2$.

Proof: Since $f \in S_0((0,\infty);X)$ vanishes on the boundaries of $(0,\infty)$, integrating by parts yields

$$S_\rho(L_k^\rho f)(\lambda) = \int_0^\infty \frac{1}{(\lambda+s)^\rho} L_k^\rho(f)(s)ds$$
$$= \int_0^\infty l_k^\rho(\lambda,s) f(s)ds$$
$$= L_k^\rho(S_\rho(f))(\lambda),$$

where

$$l_k^\rho(\lambda,t) = \frac{(2k-1)!\,\Gamma_\rho\, t^k \lambda^{k+\rho-2}}{k!\,(k-2)!\,\Gamma_{(\rho-1)}(\lambda+t)^{2k+\rho-1}}$$
$$= c_k^\rho \frac{t^k \lambda^{k+\rho-2}}{(\lambda+t)^{2k+\rho-1}} \quad . \qquad \ldots (4.2).$$

For $\rho = 1$ the results coinsides with results given in [7]. If f is defined on $(0,\infty)$ and $\bar{f}$ denotes the function $\bar{f}$ (x) $=f(e^x)$ defined on $(-\infty,\infty)$, then f→ $\bar{f}$ defines on isomorphic isomorphism between $L^p(((0,\infty);X),dx)$ and $L^p(((-\infty,\infty);X),e^x dx)$ $1 \le p \le \infty$.

If $S_\rho(x) = \frac{1}{(1+e^{-x})}$,then the generalized Stieltjes transform is defined by the kernel $S_\rho(x-y)$;

that is

$$\overline{F(\bar{u})}(x) = (S_\rho * \bar{u})(x) = \int_{-\infty}^{\infty} S_\rho(x-y)\bar{u}(y)dy$$

whenever $u \in L^p((0,\infty);X), p > 1.$

For every $u \in L^p((0,\infty);X), p > 1$, we have

$$L_k^\rho(S(u))(\lambda) = c_k^\rho \int_0^{+\infty} \frac{s^k \lambda^{k+\rho-2}}{(\lambda+s)^{2k+\rho-1}} u(s)ds$$

Where $c_k^\rho = \frac{(2k\ \ 1)!\,\Gamma_\rho}{k!\,(k-1)!\,\Gamma_{(\rho-1)}}$

Substituting $\lambda = e^x$ and s=1 in (5.4.2)

$$L_k^\rho(\overline{S_\rho}(\bar{u}))(x) = \int_{\infty}^{+\infty} \tau_k^\rho(x-y)\bar{u}(y)dy$$

where

$$\tau_k^\rho(x) = \frac{c_k^\rho e^{-x(k+1)}}{(1+e^{-x})^{2k+\rho-1}} \text{ , that is}$$
$$L_k^\rho(\overline{S_\rho}(\bar{u}))(x) = (\tau_k^\rho * \bar{u})\,(x) \qquad \ldots (4.3).$$

and for every $-\infty < x < \infty$.

From following Lemma and 4.3, we obtain the inversion formula.

Lemma 4.1: The sequence of function $(\tau_k^\rho(t)) = \frac{c_k^\rho e^{-t(k+1)}}{(1+e^{-x})^{2k+\rho-1}}$ is an approximate identity on $L^p, 1 < p < \infty$.

Theorem 4.1: Let $u \in L^p((0,\infty);X), p > 1$ and suppose λ is a point of continuity of u. Then

$$\text{Lim}_{k\to\infty} L_k^\rho(S_\rho(u))(\lambda) = u(\lambda)$$

Proof: Since $\tau_k^\rho(t)$ is an approximate identity it follows from (5.4.3) that

$$\|(\tau_k^\rho * \bar{u})(s) - \bar{u}(s)\|x \to 0 \text{ as } k \to \infty.$$

Corollary 4.1 Let $f \in L^p((0,\infty);X)$. Then

$$\mathrm{Lim}_{k\to\infty} S_\rho(L_k^\rho(f))(\lambda) = f(\lambda),\ \lambda > 0$$

5. A representation theorem for the Generalized Stieltjes transform

In this section we characterize those functions which are in the range of the generalized Stieltjes transform by determining as in [12] those vector-valued functions. $u: (0, +\infty) \to X$ for which

$$\lim_{k\to\infty} \int_0^\infty \frac{L_k^\rho(u)(s)}{(\lambda+s)^\rho} ds = u(\lambda) \qquad \text{a.e.}$$

Theorem 5.1: Let $M > o$, let $1 < p \leq \infty$ and let $f \in S_0([0,\infty); X)$. Then the following statements are equivalent.

(i) $\sup_{k>1} \|L_k^\rho(f)\|_p \leq M$

(ii)There exists a function $F \in V_p([0,\infty); X)$ such that

$$\|F\|_{V_p([0,\infty);X)} \leq M$$

and $$f(\lambda) - \int_0^\infty \frac{1}{(\lambda+s)^\rho} dF(s)\ for\ all\ \lambda, s > 0$$

Proof: Suppose (i) holds. Define the sequence of bounded operators $T_k: L^q \to X$ by

$$T_k^\rho g = \int_0^\infty L_k^\rho(f)(s) g(s) ds$$

Then by Holder's inequality

$$\|T_k^\rho g\| \leq \|L_k^\rho(f)\|_p \|g\|_q \leq M\|g\|_q$$

By corollary 4.1 for all $\lambda > 0, T_k^\rho(\varphi_\lambda) \to f(\lambda), as\ k \to \infty$

where $\varphi_\lambda^\rho(s) = \frac{1}{(\lambda+s)^\rho}$. Now the set of functions

$\{\varphi_\lambda\}_{\lambda>0}$ is total in L^q for $q > 1$, so by the uniform boundedness principle, there is a bounded operator $T^\rho: L^q \to X$ with $\|T^\rho\| \leq M$ and such that $\lim_{k\to\infty} T_k^\rho g = T^\rho g$ for all $g \in L^q$. In particular

$$f(\lambda) = \lim_{k\to\infty} T_k^\rho(\varphi_\lambda) = T^\rho(\varphi_\lambda)$$

By theorem 2.1 there exists $F \in V_p([0,\infty); X)$ such that $\|F\|_{V_p(x)} = \|T^\rho\|$ and

$Tf = \int_0^\infty f(t) dF(t)$ for all $f \in L^q$ thus $f(\lambda) = T^\rho(\varphi_\lambda) = \int_0^\infty \frac{1}{(\lambda+s)^\rho} dF(s)$

Now assume (ii), and let $F \in V_p([0,\infty); X)$ be such that

$\|F\|_{V_p([0,\infty);X)} \leq M$ and

$$f(\lambda) = \int_0^\infty \frac{1}{(\lambda+s)^\rho} dF(s)$$

Then ,

$$L_k^\rho(f)(\lambda) = \int_0^\infty l_k^\rho(\lambda, t) dF(t) \text{ where}$$

$l_k(\lambda, t)$ is defined in equation (4.2) so

$$\|L_k^\rho(\lambda,t)\| = \sup_{\|x'\|=1} |< L_k^\rho(f)(\lambda), x' >|$$

$$= \sup_{\|x'\|=1} \left|\int_0^\infty l_k^\rho(\lambda,t) < dF(t), x' >\right|$$

$$= \sup_{\|x'\|=1} \left| \lim_{|\Gamma|\to 0} \sum_{i=1}^{n} l_k^{\rho}(\lambda, t_i^*) < F(t_i) - F(t_{i-1}), x' > \right|$$
$$\leq \sup_{\|x'\|=1} \lim_{|\Gamma|\to 0} \sum_{i=1}^{n} l_k^{\rho}(\lambda, t_i^*) \|F(t_i) - F(t_{i-1})\| \|x'\|$$
$$= \lim_{|\Gamma|\to 0} \sum_{i=1}^{n} l_k^{\rho}(\lambda, t_i^*) \|F(t_i) - F(t_{i-1})\|$$
$$= \lim_{|\Gamma|\to 0} \sum_{i=1}^{n} l_k^{\rho^{\frac{1}{q}}}(\lambda, t_i^*) (t_i - t_{i-1})^{\frac{1}{q}} l_k^{\rho^{\frac{1}{p}}}(\lambda, t_i^*) \frac{\|F(t_i) - F(t_{i-1})\|}{(t_i - t_{i-1})^{\frac{1}{q}}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Thus, by applying Holder's inequality we obtain

$$\|L_k^{\rho}(f)(\lambda)\| \leq \lim_{|\Gamma|\to 0} \left(\sum_{i=1}^{n} l_k^{\rho}(\lambda, t_i^*)(t_i - t_{i-1}) \right)^{\frac{1}{q}} \times \left(\sum_{i=1}^{n} \frac{l_k^{\rho}(\lambda, t_i^*) \|F(t_i) - F(t_{i-1})\|^p}{(t_i - t_{i-1})^{\frac{p}{q}}} \right)^{\frac{1}{p}}$$
$$= \left(\int_0^{\infty} l_k^{\rho}(\lambda, t)\, dt \right)^{\frac{1}{q}} \lim_{|\Gamma|\to 0} \left(\sum_{i=1}^{n} \frac{l_k^{\rho}(\lambda, t_i^*) \|F(t_i) - F(t_{i-1})\|^p}{(t_i - t_{i-1})^{\frac{p}{q}}} \right)^{\frac{1}{p}}$$
$$\leq \lim_{|\Gamma|\to 0} \sum_{i-1}^{n} \frac{l_k^{\rho}(\lambda, t_i^*) \|F(t_i) - F(t_{i-1})\|^p}{(t_i - t_{i-1})^{p-1}}$$

Since $\int_0^1 l_k^{\rho}(\lambda, t)\, dt = 1$. By Fatou's Lemma we get

$$\int_0^{\infty} \|L_k^{\rho}(f)(\lambda)\|^p\, d\lambda \leq \int_0^{\infty} \underline{\lim}_{|\Gamma|\to 0} \sum_{i-1}^{n} \frac{l_k^{\rho}(\lambda, t_i^*) \|F(t_i) - F(t_{i-1})\|^p}{(t_i - t_{i-1})^{p-1}}\, d\lambda$$
$$\leq \underline{\lim}_{|\Gamma|\to 0} \sum_{i=1}^{n} \int_0^{\infty} l_k^{\rho}(\lambda, t_i^*)\, d\lambda \frac{\|F(t_i) - F(t_{i-1})\|}{(t_i - t_{i-1})^{p-1}}$$
$$\leq \lim_{|\Gamma|\to 0} \sum_{i=1}^{n} \frac{\|F(t_i) - F(t_{i-1})\|^p}{(t_i - t_{i-1})^{p-1}}$$
$$\leq \sup_{\Gamma} \sum_{i=1}^{n} \left(\frac{\|F(t_i) - F(t_{i-1})\|^p}{(t_i - t_{i-1})^{p-1}} \right) = \|F\|_{V_p([0,\infty);X)}^p$$

Consequently $\|L_k^{\rho}(f)\|_p \leq \|F\|_{V_p([0,\infty);X)}^p$ as required .

Corollary 5.1 For $p > 1$ the generalized Stieltjes transform is an isometric isomorphism between $V_p([0,\infty);X)$ and $W^p([0,\infty);X)$ where

$$W^p([0,\infty);X) = \left\{ f \in S_0((0,\infty);X) : \|f\| = \sup_k \|L_k^{\rho}(f)\|_p < \infty \right\}.$$

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