International Journal of Advanced Computer and Mathematical Sciences
ISSN 2230-9624. Vol 5, Issue1, 2014, pp9-13
http://bipublication.com

MAXIMUM INDEPENDENT SETS AND WELL COVERED GRAPHS

Dinesh K. Thakkar[1], Ashish A. Prajapati[2] and Jitendra. V. Changela[3]

[1]Department of Mathematics,
Saurashtra University Campus, Rajkot – 360 005, Indina.
[2]Mathematics Department,
Government Engineering Collage, Jhalod Road, Dahod – 389 151. India.
[3]Mathematics Department,
Parul Institute of Technology, Limda, Ta. Vagodia – 391 760. India.
[*]Corresponding author: Email: dkthakkar1@yahoo.co.in; ashishprajapati14@gmail.com

[Received-20/02/2014, Accepted-01/03/2014]

ABSTRACT:

In this paper we consider maximum independent sets and independence number of a graph. We prove a necessary and sufficient condition under which number decreases when a vertex is removed from graphs. We also prove that if G is a vertex transitive graph and v is a vertex of G such that removal of v decreases the independence number of G. Then all other vertices are also like v. We also consider well covered and approximately well covered graphs, and deduce some results regarding the independence number of these graphs.

Keywords: Independent set, Maximal independent set, Maximum independent set, Vertex transitive graph, well covered graph, Approximately well covered graph.

AMS Subject Classification (2010): 05C69, 05C50, 05C99.

[I] INTRODUCTION:

The independence number of a graph is the cardinality of a maximum independent set of a graph. The cardinality of such a set is denoted as $\beta_0(G)$. When a vertex is removed from the graph G, the independence number of the graph may increase, decrease or remain unchanged. In this paper we prove that these numbers cannot increase and further we prove a necessary and sufficient condition under which these number decreases. Further we consider vertex transitive graph and prove that all vertices are of the same type. We also consider well covered graphs and approximately well covered graphs.

[II] MACIMUM INDEPENDENCE:

Definition: 2.1 [5]

Let G be a graph and S be a subset of $V(G)$, then S is said to be an independent set, if any two distinct vertices of S are non adjacent.

Definition: 2.2[5]
An independent set S is said to be a maximal independent if it is not properly contain in any independent set.

Definition: 2.3 [5]
An independent set of maximum size is called a maximum independent set.

Remark
Note that every maximum independent set is a maximal independent set but the converse is not true.
Also note that every maximal independent set is an independent dominating set.
Also note that the complement of an independent set is a vertex covering set. Therefore a set S is a maximal independent if and only if $V(G) - S$ is a minimal vertex covering set. Also a set S is maximum independent if and only if $V(G) - S$ is a minimum vertex covering set.

Definition: 2.4 [5]
The cardinality of a maximum independent set is called the independence number of the graph G and it is denoted as $\beta_0(G)$.
Thus, if a graph has n vertices then $\alpha_0(G) + \beta_0(G) = n$.

[III] MAIN RESULTS:
Now we prove the following theorem.

Theorem: 3.1 If G is a graph and $v \in V(G)$ then $\beta_0(G - v) \leq \beta_0(G)$.

Proof: Let S be a maximum independent set of $G - v$. If v is not adjacent to any vertex of S then $S \cup \{v\}$ is an independent set in the graph G.
Therefore, $\beta_0(G - v) < |S \cup \{v\}| \leq \beta_0(G)$.
If v is adjacent to same vertex of S, then S is a maximal independent set in the graph G. Therefore, $\beta_0(G - v) < |S| \leq \beta_0(G)$.

Thus, in all the cases $\beta_0(G - v) \leq \beta_0(G)$
Now we introduce the following notations.

$I^- = \{v \in V(G) : \beta_0(G - v) < \beta_0(G)\}$

$I^0 = \{v \in V(G) : \beta_0(G - v) = \beta_0(G)\}$

First we prove the following lemma.

Lemma: 3.2 If $v \in I^-$ then $\beta_0(G - v) = \beta_0(G) - 1$.

Proof: We know that $\beta_0(G - v) < \beta_0(G)$. Let T be a maximum independent set of G, then $\beta_0(G - v) < |T|$.
If $v \notin T$ then T is a maximum independent set of $G - v$, which is not possible. Therefore, $v \in T$. Now $T - v$ is an independent set in $G - v$ and its size must be maximum, because $v \in I^-$.
Therefore,
$\beta_0(G - v) < |T - v| = |T| - 1 = \beta_0(G) - 1$

Theorem: 3.3 Let G be a graph then $v \in I^0$ if and only if there is a maximum independent set S of G such that $v \notin S$.

Proof: Suppose $v \in I^0$. Let S be a maximum independent set of $G - v$, then S is also a maximum independent set of G, because $v \in I^0$. Obviously, $v \notin S$.
Conversely, suppose there is a maximum independent set S of G such that $v \notin S$. Then S is also a maximum independent set of $G - v$. Therefore, $v \notin I^0$

Corollary: 3.4 If $v \in I^-$ if and only if v belongs to every maximum independent set of G

Corollary: 3.5 I^- is equal to the intersection of all maximum independent sets of G

Corollary: 3.6 Let G be a graph then $I^- = V(G)$ if and only if the graph G is a null graph.

Proof: Suppose G is a null graph then G has only one maximum independent set. Therefore $I^- = V(G)$.

Conversely, suppose $I^- = V(G)$.

Since, I^- is contained in every maximum independent set, $V(G)$ is contained in every maximum independent set. Therefore G has only one maximum independent set namely $V(G)$.

Hence G is a null graph

Corollary: 3.7 A graph G has at least one edge if and only if I^0 is a non empty set

Corollary: 3.8 If u and v belongs to I^- then u and v are non adjacent.

Proof: Let S be a maximum independent set of G then u and v belongs to S. Since S is an independent, u and v are non adjacent

Corollary: 3.9 Let G be a graph and v be a vertex of G then if $v \in I^-$ then all its neighbors are in I^0.

Proof: Suppose u is a neighbor of v. Let S be a maximum independent set of G, then $v \in S$ and $u \notin S$. Thus by previous theorem $u \in I^0$.

In other words $N(v)$ is a subset of I^0

Corollary: 3.10 For any graph G, $\delta(G) \leq |I^0|$

Now we consider vertex transitive graphs. In these graphs there are enough automorphisms.

Definition: 3.11

Let G be a graph then G is said to be a vertex transitive graph if for ant two vertices u and v of G, there is an automorphism $f: V(G) \to V(G)$ such that $f(v) = u$.

The complete graph K_n, the cycle C_n and the Peterson graphs are some examples of vertex transitive graphs.

We now prove the following theorem.

Theorem: 3.12 Let G be a vertex transitive graph and $v \in V(G)$ such that $v \in I^0$ then every vertex of G is a member of I^0. That is $I^0 = V(G)$.

Proof: Since $v \in I^0$ there is a maximum independent set S such that v does not belongs to S.

Let u be any vertex of G, then there is an automorphism $f: V(G) \to V(G)$ such that $f(v) = u$.

Now $f(S)$ is a maximum independent set because f is an automorphism. Since v does not belongs to S, $f(v)$ does not belongs to $f(S)$. That is u does not belongs to $f(S)$. Thus there is a maximum independent set namely $f(S)$ which does not contain u. Therefore, $u \in I^0$

Corollary: 3.13 Let G be a vertex transitive graph and $v \in V(G)$. If $v \in I^-$ then every vertex of G belongs to I^-. That is $I^- = V(G)$.

Proof: Let u be any vertex of G, then there is an automorphism f such that $f(v) = u$. Since v belongs to every maximum independent set of G, $f(v)$ belongs to every maximum independent set of G. That is u belongs to every maximum independent set of G. Hence $u \in I^-$

Theorem: 3.14 Let G be a vertex transitive graph then the union of all maximum independent sets of G is $V(G)$.

Proof: Suppose u is a vertex which does not belongs to any maximum independent set of G and $v \in S$. Now there is an automorphism f of G such that $f(v) = u$. Since $v \in S$,

$f(v) \in f(S)$ and $f(S)$ is a maximum independent set, which contains u. This contradicts our assumption. Therefore, union of all maximum independent sets is $\boldsymbol{V(G)}$.

Example: 3.15 Consider the path graph $G = P_5$. Whose vertices are 1, 2, 3, 4, and 5.

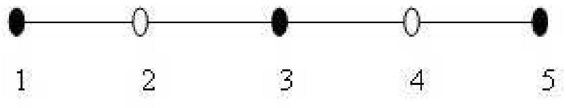

Figure: Path graph with five vertices

This graph has only one maximum independent set $S = \{1,3,5\}$ and the union of maximum independent set is not $V(G)$. This is because the path graph P_5 is not vertex transitive. In fact it is not even regular graph.

[IV] WELL COVERED GRAPHS:

$\alpha_0(G) = \textit{The size of the smallest vertex covering set of } G$

$i(G) = \textit{The independent domination number of } G$

$= \textit{The size of the smallest maximal independent set}$

$\Gamma_{cr}(G) = \textit{The size of the largest minimal vertex covering set of } G$

$V_i^+ = \{v \in V(G) : i(G - v) > i(G)\}$

$V_i^- = \{v \in V(G) : i(G - v) < i(G)\}$

$V_i^0 = \{v \in V(G) : i(G - v) = i(G)\}$

Note that,

I.

et is a vertex covering set if and only if its compliment is an independent set.

II.

set is minimal vertex covering set if and only if its compliment is a maximal independent set.

III.

is well covered if and only if $i(G) = \beta_0(G)$.

We will denotes maximal independent set with minimum cardinality as an $i - set$ of G.

Note that a graph G is well covered if and only if all maximal independent sets have the same cardinality; equivalently all independent dominating sets have same cardinality.

Definition: 4.1 [4]

A graph G is said to be a well covered if any two minimal vertex covering sets have the same cardinality.

Equivalently a graph G is well covered if $\alpha_0(G) = \Gamma_{cr}(G)$.

Definition: 4.2

A graph G is said to be an approximately well covered graph if $\alpha_0(G) = \Gamma_{cr}(G) - 1$.

For example Peterson graph is an approximately well covered graph.

Here we consider well covered graphs. We use the following notations.

Theorem: 4.3 Let G be a graph and $v \in V(G)$. Then

1)

f G is well covered then V_i^+ is empty.

2)

f $v \in V_i^-$ and G is well covered then either $v \in I^0$ or $G - v$ is well covered.

3)

f G is well covered and $v \in V_i^0$ then $G - v$ is well covered and $v \in I^0$.

Proof: 1) If there is a vertex v in V_i^+ then

$i(G) < i(G - v) \leq \beta_0(G - v) \leq \beta_0(G)$. s

Since $i(G) = \beta_0(G)$,

$i(G - v) = \beta_0(G) = i(G)$. A

Which contradicts the fact that $i(G - v) \geq i(G)$. Hence, V_i^+ is empty.

2) Now $i(G-v) < i(G) = \beta_0(G)$. Also, $i(G-v) \le \beta_0(G-v) \le \beta_0(G)$. If $\beta_0(G-v) = \beta_0(G)$ then $v \in I^0$.

Otherwise, $\beta_0(G-v) = i(G-v)$, which implies that $G-v$ is well covered.

3) Now, $i(G-v) = i(G) = \beta_0(G)$. Also,

$i(G-v) \le \beta_0(G-v) \le \beta_0(G) = i(G)$.

Since $i(G-v) = i(G)$, which implies that $\beta_0(G-v) = i(G-v) = i(G) = \beta_0(G)$. Thus, $G-v$ is well covered and $v \in I^0$.

Also we consider so called an approximately well covered graphs which have been already defined earlier.

Note that a graph G is approximately well covered if and only if, $i(G) = \beta_0(G) - 1$. (Because $\alpha_0(G) = \Gamma_{cr}(G) - 1$ implies $n - \alpha_0(G) = n - \Gamma_{cr}(G) + 1$. That is $\beta_0(G) = i(G) + 1$)

Theorem: 4.4

1)

uppose G is approximately well covered and $v \in V_i^+$ then $i(G-v) = i(G) + 1$, and $v \in I^0$, and $G-v$ is well covered.

2)

f G is approximately well covered and $v \in V_i^-$ then either $\beta_0(G-v) = \beta_0(G) - 2$, and $v \in I^-$, or $\beta_0(G-v) = \beta_0(G) - 1$, and $v \in I^-$, or $v \in I^0$.

3)

f G is approximately well covered and $v \in V_i^0$ then either $v \in I^-$, and $G-v$ is well covered or $v \in I^0$.

Proof: 1) Suppose G is approximately well covered and $v \in V_i^+$.

Then

$i(G) < i(G-v) \le \beta_0(G-v) \le i(G) + 1$.

This implies that $i(G-v) = i(G) + 1$, and $\beta_0(G-v) = i(G-v)$ and thus $G-v$ is well covered and since $\beta_0(G-v) = i(G) + 1 = \beta_0(G)$,

$v \in I^0$.

2) $i(G-v) = i(G) - 1$,

$i(G) - 1 \le \beta_0(G-v) \le i(G) + 1$.

If $\beta_0(G-v) = i(G) + 1$.

Then $\beta_0(G-v) = \beta_0(G) - 2$, and hence $v \in I^-$.

If $\beta_0(G-v) = i(G) = \beta_0(G) - 1$, then $v \in I^-$. If $\beta_0(G-v) = i(G) + 1$, then $\beta_0(G-v) = \beta_0(G)$ and hence $v \in I^0$.

3) $i(G-v) = i(G) \le \beta_0(G-v) \le \beta_0(G)$.

If $\beta_0(G-v) = i(G-v)$, then $G-v$ is well covered and $v \in I^-$. Otherwise $\beta_0(G-v) = i(G-v)$ and $v \in I^0$

S

REFERENCES:

[1] A. Finbow, B. Hartnell and R. Nowakowski, "*Well Dominated Graphs: a collection of well covered ones.*" Ars Combin, 25(1988), pg 5 – 10.

I

[2] D. K. Thakkar, B. M. Kakrecha, G. J. Vala "*Well Dominated and Approximately Well Dominated Graphs*" Discrete Mathematics, Elixir Dis. Math. 53(2012) 11973-11975.

[3] D. K. Thakkar and J. C. Bosamiya "*Vertex Covering Number of a Graph*" Mathematics Today, Vol. 27, June 2011, pg 30 – 35.

I

[4] T. W. Hynes, S. T. Hedetniemi, P. J. Slater "*Fundamental of Domination in Graphs*", 1998, Marcel Dekkar. Inc.

[5] T. W. Hynes, S. T. Hedetniemi, P. J. Slater "*Advanced Topics on Domination in Graphs*", 1998, Marcel Dekkar. Inc.