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A COMPARISON OF VARIOUS UPPER BOUNDS ON $A_2(7,4)$

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ABSTRACT: -

Van Pul [6] gave the detail configuration of upper bounds. Lindel [3] and Williams [4] clarify the theory of error correcting codes. In this paper we have taken a $A_2(7,4)$ code and various type of upper bounds are applied on it and comparison of these have been done and the best method for upper bounds is discussed.

Keywords: Singleton Upper Bound, Elias Upper Bound ,Hamming or sphere-packing bound, Plotkin bound, Johnson Upper Bound

INTRODUCTION:

The upper bounds of the code is the size (or rate) of the code as a function of their relative distances. All the bounds apply for general codes and they do not take advantage of linearity. The Singleton Upper Bound, Elias Upper Bound ,Hamming or sphere-packing bound, Plotkin bound, Johnson Upper Bound gave an upper bound on the size (or rate) of codes, which is our focus in this paper. A generalization of Hamming codes called binary BCH codes, when d a fixed constant is and the block length is allowed to grow, the Hamming bound is again tight up to lesser order terms. Now we are defining different types of bounds.

Definition of $A_q(n,d)$:- For given length and number of codewords, a fundamental problem in coding theory is to produce a code with largest possible 'd'.Alternatively,given n and ,determine the maximum number i.e., $A_q(n,d)$.It denotes the size of the largest q-ary code of block length n and distance d. We denote by $A_q(n,d,w)$ the size of a largest constant weight code of block length n and distance d. all of whose codewords have Hamming weight w.

Let C be a $(n,M,d)_q$ code. This notation means that $C \subset F_q^n$, possibly nonlinear, the number of codewords is M, and

$d(x,y) \geq d, \forall x,y \in C, x \neq y$ where there exists a $x,y \in C$ with $d(x,y) = d$.

If C is a $(n,k,d)_q$ code, then C is an $(n,q^k,d)_q$ code.

Notation:

$A_q(n,d) = \max\{M : \exists (n,M,\geq d)_q\}$

$B_q(n,d) = \max\{q^k : \exists (n,k,d)_q\}$ clearly

$B_q(n,d) \leq A_q(n,d)$

Theorem 1: Assume $d \geq 1$ Then,

1. For $d = 1, B_q(n,1) = q^n$ **and** $A_q(n,1) = q^n$
2. $A_q(n,d) \leq A_q(n-1,d-1)$
3. If $q = 2$ and d is even, $A_q(n,d) = A_q(n-1,d-1)$
4. (2), (3) and (5) hold if we replace all $\boldsymbol{A}$'s with $\boldsymbol{B}$'s.
5. If $q = 2$ and d is even, set $M = A_2(n,d)$ Then there exists $(n,M,\geq d)$ code such that $w(x)$ is even for all $x \in C$ and (x,y) is even for all $x,y \in C$.

Singleton Upper bound: This is the simplest among all the bounds. The family of codes which meet the Singleton bound are called maximum distance separable (MDS) codes. However, Reed-Solomon and other MDS codes will be (necessarily) defined over an alphabet that grows with the block length. For code families over a fixed alphabet such as binary codes, substantial improvements to the Singleton bound are possible and it is defined as

Singleton Bound: $A_q(n,d) \leq q^{n-d+1}$ for $d \leq n$

Proof :- If $d = n$, $A_q(n,d) = q = q^{n-n+1}$.and If $d < n$, then

$$A_q(n,d) \leq qA_q(n-1,d) \leq q^2 A_q(n-2,d) \leq \ldots\ldots \leq q^{n-d} A_q(d,d) = q^{n-d+1}$$

Sphere Packing Bound: The fact that the sphere of radius t about codeword are pair wise disjoint immediately implies the following elementary inequality called Sphere Packing Bound or Hamming Bound.

$$A_q(n,d) \leq \frac{q^n}{\sum_{i=0}^{t} \binom{n}{i} (q-1)^i}$$ Where $t = \frac{d-1}{2}$. If d is minimum distance of a code c and $t = \frac{d-1}{2}$, then sphere of radius t about distinct codewords are disjoints.

Plotkin Bound: This bound is an upper bound which often improves the sphere packing bound on $A_q(n,d)$. However, it is only valued when d is sufficient close to n.Let C be an (n,M,d) code over F_q^n such that $rn < d$ where $r = 1 - q^{-1}$, then

$$M \leq \left\lfloor \frac{d}{d-rn} \right\rfloor,$$

In particular cases $A_q(n,d) \leq \left\lfloor \frac{d}{d-rn} \right\rfloor$,For binary case i.e, for q=2 it is $A_2(n,d) \leq 2\left\lfloor \frac{d}{2d-n} \right\rfloor$, if $n < 2d$.

Elias Upper Bound: The Elias bound for the finite case and the asymptotic case are given by the following result. Let $r,n,d,w \in N, q \geq 2, \theta = 1 - q^{-1}$. and assume that $w \leq rn$ and $w^2 - 2rnw + rnd > 0$,then

$$A_q(n,d) \leq \frac{rnd}{w^2 - 2rnw + rnd} \frac{q^n}{V_q(n,w)}$$

Where $V_q(n,w) = \sum_{i=0}^{w} \binom{n}{i} (q-1)^i$

Johnson upper Bound: Johnson [1] has improved the sphere packing bound. Johnson used the quantity $A(n,d,w)$, which is the maximum number of codewords in a binary code of length n, constant weight w, and minimum distance d

$$A_q(n,d) \le \frac{nd(q-1)}{qw^2 - 2(q-1)nw + nd(q-1)}, Deno > 0.$$

For $q = 2$,

$$A_2(n,d) \le \frac{nd}{2w^2 - 2nw + nd}, Deno > 0.$$

Calculations: In this section we considered an $A_2(7,4)$ and applied different types of bounds on it

(i) Singleton Bound: For $A_2(7,4)$, we have $n = 7, d = 4$

$$A_2(7,4) \le 2^{7-4+1}$$
$$\le 2^4$$
$$\le 16$$

(ii) Sphere Packing Bound: For $A_2(7,4)$, we have $n = 7, d = 4$

$$A_2(7,4) \le \frac{2^7}{\sum_{i=0}^{1} \binom{7}{i}(2-1)^i}$$
$$\le \frac{128}{\binom{7}{0}(2-1)^0 + \binom{7}{1}(2-1)1}$$
$$\le 16$$

But by using result of Theorem 1 (2),

$$A_2(7,4) \le A_2(6,3) \le 9$$
$$so, A_2(7,4) \le 9.$$

(iii) Plotkin Bound: For $A_2(7,4)$, we have $n = 7, d = 4$

$$A_2(7,4) \le$$
$$\le 2\left\lfloor \frac{4}{2(4)-7} \right\rfloor$$
$$\le 2\left\lfloor \frac{4}{8-7} \right\rfloor$$
$$\le 8$$

(iv) Elias Upper Bound: For $A_2(7,4)$, we have

$$n = 7, d = 4, r = 1 - 2^{-1} = 1/2,$$
$$w \le 1/2(7) \le \frac{7}{2} \le 3.$$

So, $w = 0,1,2,3.$ Also

$$V_2(7,w) = \sum_{i=0}^{w} \binom{7}{i}(2-1)^i$$
$$\therefore V_2(7,0) = 1, V_2(7,1) = 7,$$
$$V_2(7,2) = 21, V_2(7,3) = 35.$$

By using above results, we get following results

For
$w = 0 \Rightarrow A_2(7,4) \le 128$
$w = 1 \Rightarrow A_2(7,4) \le 32$
$w = 2 \Rightarrow A_2(7,4) \le 21$

(v) Johnson Upper Bound: $A_2(7,4)$, we have $n = 7, d = 4, d \le 2w \Rightarrow w \ge \frac{d}{2} = 2.$ So,

$$A_2(7,4,2) \le \frac{7(4)}{2(4) - 2(14) + 28}$$
$$\le 3.5$$
$$\le 3$$

Comparison and Conclusion: - Agrell [5] have discussed all the upper bound and we have also applied these upper bounds on a single code and we conclude following statement

Bounds	$A_2(7,4) \le$	W=0	W=1	W=2
Singlenton Bound	16			
Sphere packing Bound	9			
Plotkin Bound	8			
Elias Bound		128	32	21
Johnson Upper Bound				03

According to our results,it have been observed that Plotkin bond is the tight bound but when we use the concept of weight than Johnson upper bound is the tight one.

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