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On The Convergence of Sequence of Attractors in the Fractal Space

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ABSTRACT

In this paper, we discuss the topology induced by Hausdorff metric on the set of all non empty compact subsets of a complete metric space. We examine the relation between this topology and the convergence of sets. We also prove the convergence of a sequence of sets which are attractors to a converging sequence of Iterated Function Systems. We have also proved the convergence in the case of Countable Iterated Function Systems and Generalised Iterated Function Systems.

Keywords: Topology, Iterated Function System, Convergence, Sequence

1. INTRODUCTION

Here we discuss the ideal space to study fractal geometry. We work in a complete metric space (X, d) and then consider the set of all compact subsets of X with Hausdorff metric. Informally, the Hausdorff metric, named after Felix Hausdorff, measures the largest length out of the set of all distances between each point of a set to the closest point of a second set. Given a metric space, the Hausdorff metric induces a topology on the space of all compact subsets of the metric space. This space is referred to as the "space of fractals" or the fractal space. We investigate some interesting properties of this topology in the first section.

Iterated function systems (IFS) are one of the most common and general ways to generate fractals. The notion of IFS was introduced by Hutchinson [6] and was popularised by Barnsley [10]. These methods are useful tools to build fractals and other self similar sets as attractors of dynamical systems. Fractal methods are quite popular in the modelling of natural phenomena in computer graphics, engineering sciences, physics and so forth.

In this paper, we discuss some results on the convergence of the attractors of the IFS, for both finite and infinite case. We have also seen the convergence in the case of generalisation of IFS which are contractions from X^m to X.

2. HAUSDORFF METRIC TOPOLOGY

Let (X, d) be a complete metric space. Consider K(X), the set of all non-empty compact subsets of X. We define a metric h on K(X) as follows:

$$h(A, B) = \max\{\sup_{x\in A}\{\mathrm{dist}(x, B)\}, \sup_{y\in B}\{\mathrm{dist}(y, A)\}\}$$

for all subsets $A, B \in K(X)$. Here h is called the Hausdorff metric on K(X).

Consider the topology on K = K(X) U φ induced by the Hausdorff metric h called the Hausdorff metric topology.

The collection of all ε-balls $Bh(A, \varepsilon)$ for A ∈ K and ε > 0, forms a basis for the Hausdorff metric topology T_h. Here $Bh(A,\varepsilon) = \{B / h(A,B) < \varepsilon\}$.

The metric h preserves the metric on X. ie; $h(\{x\},\{y\}) = d(x,y)$, for all x, y ∈ X.

Analogous to the definitions and results from the classical analysis, we make the following definitions.

Definition 2.1: A sequence $(A_1, A_2, \ldots\ldots)$ of subsets of K(X) is said to converge to a compact set A iff for every neighbourhood $\{U\}$ of A, there exist a positive integer N such that $A_i \in \{U\}$, for all $i \geq N$ and we say $\{A_n\}$ converges to A, ie; $A_n \to A$.

Definition 2.2: A sequence $(A_1, A_2, \ldots\ldots)$ of subsets of K(X) is said to be a Cauchy sequence in K(X), if there exist a positive integer N such that for all $m,n > N, h(A_m, A_n) < \varepsilon$, for a given $\varepsilon > 0$.

Definition 2.3: The Hausdorff metric space (K, h) is complete if every Cauchy sequence $\{A_n\}$ is complete in K.

Theorem 2.4

If the metric space (X,d) is complete, (K, h) is complete.

Proof can be found in Barnsley [10].

Theorem 2.5

If (X,d) is totally bounded, (K, h) is totally bounded.

Proof: Fix ε > 0.

We have X totally bounded. Let F be a finite subset of X with $X \subset B_h(F,\varepsilon)$.

For each compact subset A of X, there exist a minimal subset E of F such that $A \subset B_h(E,\varepsilon)$

And it follows that the reverse inclusion $E \subset B_h(A,\varepsilon)$ must be satisfied. As a result, each compact subset of X has Hausdorff distance at most ε from some subset of F. Since F has only finitely many subsets and ε > 0 was arbitrary, (K, h) is totally bounded.

Theorem 2.6

If (X,d) is a compact metric space, (K, h) is compact.

Proof: A metric space is compact iff it is complete and totally bounded. Hence the theorem follows from Theorem 2.2 and 2.3.

Similar proofs on the set of all closed subsets of X can be found in Beer [5] and Michael [3].

3. Convergence of sequences

In this section, we discuss the convergence of sequence of sets in the Hausdorff distance. Before going to the main results, we will see some of the basic definitions and results from Falconer [7, 8] required for the understanding of the main results.

Definition 3.1: A mapping $f: X \to X$ is a contraction if there is a number c with $0 < c < 1$ such that $d(f(x), f(y)) \leq c.d(x,y)$ for all $x, y \in X$. Clearly, every contraction is a continuous mapping.

Definition 3.2 Let $\{f_n\}$ be a sequence of contractions and c_n is the contraction factor of f_n for each $n \in N$.Then the contraction factor of the sequence is defined as $c = \max\{c_n, n \in N\}$.

Definition 3.3: A hyperbolic iterated function system (IFS) consists of a complete metric space together with a finite set of contraction mappings $f_j: X \to X$ with respect to the contraction factors $c_j, j = 1, \ldots N$. The contraction factor of the IFS is defined as $c = \max\{c_j, j = 1, \ldots, N\}$. The notation for the IFS just defined is $\{X; f_j, j = 1, \ldots, N\}$.

Then the system $(f_1, \ldots, f_N)$ generates a natural contraction mapping F on K(X) with contraction factor c: $F(B) = \bigcup_{i=1}^{N} f_i(B)$. By Banach fixed point theorem, there exist a unique set $A \in$ K(X) such that $F(A) = A$, and for any $B \in$ K(X), the sequence of iterates $F^n(B)$ converges to A. The unique fixed point A is called the attractor of the IFS $\{X; f_j, j = 1, \ldots, N\}$.

Theorem 3.4

Let (X,d) be a complete metric space and $\{f_n\}$ be a sequence of contractions on X with contraction factor 'c' which is pointwise convergent to a contraction f with contraction factor 'c'. Let x_n be the fixed point of f_n and x be the fixed point of f. If $\{x_n\}$ is bounded. Then $x_n \to x$.

Proof: For any $n \in N$, we have

$$d(x_n, x) \leq d(f_n(x_n), f_n(x)) + d(f_n(x), f(x))$$
$$\leq cd(x_n, x) + d(f_n(x), f(x))$$

Since $\{f_n\}$ converges to f, the result follows.

The following two theorems are from Mihail [1] which will be used to prove the main results.

Theorem 3.5

Continuity Theorem: Let (X, d) be a complete metric space, f *and* g are contractions on X. let α be the fixed point of f and β be the fixed point of g. Then $d(\alpha,\beta) \le \bar{d}(f,g)\frac{1}{1-\min\{Lip\,f, Lip\,g\}}$ where $\bar{d}(f,g) = \sup_{x\in X} d(f(x),g(x))$ and $Lip\,f = \sup_{x,y\in X, x\neq y}\frac{d(f(x),f(y))}{d(x,y)}$.

Theorem 3.6

Let (X,d) be a complete metric space and $x \in X$. Let $I_1 = \{X; f_i, i = 1, \dots. n\}$ and $I_2 = \{X; g_i, i = 1, \dots. n\}$ be two IFSs with contraction factors c_1 and c_2 and attractors A_1 and A_2 respectively. Then $h(A_1,A_2) \le \max_{k=1,\dots,n} \bar{d}(f_k,g_k).\frac{1}{1-min.(c_1,c_2)}$, where $\bar{d}(f_k,g_k) = \sup_{x\in X} d(f_k(x),g_k(x))$.

The main results on the convergence of the sequence of attractors is summarised in the following two theorems.

Theorem 3.7

Let (X,d) be a complete metric space. Let $\{X; f_1,\dots,f_m\}$ be an IFS with attractor $A \in$ K(X) such that $A = \bigcup_{i=1}^{m} f_i(A)$. If there exist a sequence of contractions $\{f_{nk}\}$ which converges to $f_k, k = 1,\dots,m$. Then $\{X; f_{j1}, f_{j2},\dots,f_{jm}\}$ is an IFS having attractor A_j, for j=1, 2,....., and the sequence of sets $\{A_n\}$ converges to A.

Proof: We have a sequence $\{f_{nk}\}$ converging to $f_k, k = 1,\dots,m$, and an IFS $\{X; f_1,\dots,f_m\}$ with attractor A.

Since $\{f_{j1}, f_{j2},\dots,f_{jm}\}$ for each j is a finite set of contractions with contraction factor c_j, the IFS $\{X; f_{j1}, f_{j2},\dots,f_{jm}\}$ has an attractor A_j. Hence we get a sequence $\{A_n\}$ of sets in K(X).

Using Thoerem 3.4, for all i,j

$$h(A_i,A_j) \le \max_{k=1,\dots,m} \bar{d}(f_{ik},f_{jk}).\frac{1}{1-min.(c_i,c_j)}$$

But $\bar{d}(f_{ik},f_{jk})$ decreases, since $\{f_{nk}\}$ converges to f_k, and $\{c_n\}$ converges to, say c, the contraction factor of the IFS $\{X; f_1,\dots,f_m\}$.

Hence,

$h(A_n,A) \le \max_{k=1,\dots,m} \bar{d}(f_{nk},f).\frac{1}{1-min.(c_n,c)} \to 0$ *as* $n \to \infty$

.

ie; $\{A_n\}$ converges to A on K(X).

As in the case of above theorem, if there exist converging sequences of contraction mappings to each member of an IFS on X, we get a converging sequence of contraction mappings on K(X). This result is summarised in the following theorem.

Theorem 3.8

Let (X,d) be a complete metric space. Let $\{X; f_1,\dots,f_m\}$ be an IFS with attractor $A \in$ K(X) such that $A = \bigcup_{i=1}^{m} f_i(A)$. If there exist a sequence of contractions $\{f_{nk}\}$ which converges to $f_k, k = 1,\dots,m$. Then there exist a sequence of contraction mappings $\{F_n\}$ on K(X) which converges to F whose fixed point is A.

Proof: For each j, $\{f_{j1}, f_{j2},\dots,f_{jm}\}$ is a finite set of contractions which induces a natural mapping F_j on K(X): $F_j(B) = \bigcup_{i=1}^{N} f_{ji}(B)$.

Since F_j is a contraction mapping on a complete metric space, it has a unique fixed point A_j. Now using theorem 3.1.1, this sequence $\{A_n\}$ converges to A, which is the fixed point of a contraction mapping, say F, induced by $\{f_1,\dots,f_m\}$ the collection of contraction mappings on X. Thus the sequence $\{F_n\}$ converges to F on K(X).

3.1 Nature of Convergence

From equation (1), it is clear that the the $h(A_1,A_2)$ depends on the value of $d(f_k,g_k)$. So as the sequence $\{f_{nk}\}$ converges pointwise on $C(X,X)$, the sequence of sets $\{A_n\}$ converges pointwise to A in H(X). The case is similar for uniform convergence. Thus the Theorem 3.7 can be modified whose proof follows directly from Theorems 3.6 and 3.7 and the definition of pointwise and uniform convergence.

Theorem 3.9. Let (X,d) be a complete metric space. Let $\{X; f_1,\dots,f_m\}$ be an IFS with attractor $A \in$ K(X) such that $A = \bigcup_{i=1}^{m} f_i(A)$. If there exist a sequence of contractions $\{f_{nk}\}$ which converges pointwise/uniformly to $f_k, k = 1,\dots,m$. Then $\{X; f_{j1}, f_{j2},\dots,f_{jm}\}$ is an IFS having attractor A_j, for j=1, 2,....., and the sequence of sets $\{A_n\}$ converges pointwise/uniformly to A.

In the next section we test for convergence in the case of a generalisation of iterated function systems which consist of contractions from X^m to X.

4. Generalised iterated function system

The notion of GIFS was introduced by A. Mihail and R. Miculescu[2].

Definition 4.1: Let (X,d) be a complete metric space. A generalised iterated function system (GIFS) on X of order m, denoted by $S = (X,(f_k)k = 1,\dots,n)$ where $f_k : X^m \to X$ for $k = 1,\dots,n$ are contraction mappings.

The following theorem ensures the existence of a fixed point for a given GIFS.

Theorem 4.2: Let (X,d) be a complete metric space and $S = (X,(f_k)k = 1,\dots,n)$ a GIFS of order m with $c = \max_{k=1,2,\dots,n} Lip(f_k) < 1$. Then there exists a unique $A(S) \in H(X)$ such that $F_S(A(S),A(S),\dots,A(S)) = A(S)$.

Here $A(S)$ is the fixed point or the atttractor of the given GIFS S.

In the case of a GIFS the theorem 3.6 will take the following form.

Theorem 4.3: Let (X,d) be a complete metric space. If $S = (X,(f_k)k = 1,\dots,n)$ and $S' = (X,(g_k)k = 1,\dots,n)$are two GIFSs of order m such that $c = \max\{Lip(f_1),\dots\dots,Lip(f_n)\} < 1$ and $c' = \max Lip(g_1,\dots\dots,g_n) < 1$, then

$$h(A(S),A(S')) \leq \frac{1}{1-\min(c,c')} max\{\bar{d}(f_1,g_1),\dots,\bar{d}(f_n,g_n)\} \quad (2)$$

The convergence of sequence in the case of GIFS is summarised in the following theorem.

Theorem 4.4: Let (X, d) be a complete metric space. Let $S = (X,(f_k)k = 1,\dots,n)$ be a GIFS with the attractor $A(S) \in H(X)$ such that $F_S(A(S),A(S),\dots,A(S)) = A(S)$ where $F_S(H_1,H_2,\dots,H_m) = \cup_{k=1}^{n} F_{f_k}(H_1,H_2,\dots,H_m) = \cup_{k=1}^{n} f_k(H_1 \times H_2 \times \dots \times H_m)$ is the set function associated with the GIFS S. If there is a sequence of contractions $\{f_{ik}\}$ which converges to f_k, $k = 1,\dots,n$ where each $f_{ik}: X^m \to X$. Then $S_j = \{X; (f_{j1},\dots,f_{jn})\}$ is a GIFS having attractor $A(S_j)$ $for\ j = 1,2,\dots$ and the sequence of sets $A(S_i)$ converges to $A(S)$ in $H(X)$.

Proof. Here the sequence of mappings $\{f_{ik}\}$ are contractions from $X^m \to X$ which converges to $f_k: X^m \to X$. Then we get GIFS $S_j = \{X; (f_{j1},\dots,f_{jn})\}, j = 1,2,\dots$. Then by the theorem , there exist a unique set $A(S_j) \in H(X)$ such that $F_{S_j}(A(S_j), A(S_j),\dots,A(S_j)) = A(S_j)$ where F_{S_j} is the set function associated with the GIFS S_j . Thus we get a sequence $A(S_i)$ of sets in $H(X)$.

Now the proof runs similar to that in Theorem 3.7.

In the next section, we extend these results to the case of infinite IFS.

5. Countable iterated function system

The definition of Countable Iterated Function System (CIFS) was given by N. A. Secelean [12].

A sequence of contractions $\{f_n\}_{n\geq 1}$ with contraction ratios c_n, for $c_n > 0$, such that $0 < c_n < 1$ is called a countable iterated function system, which is denoted as CIFS.

Let $\{f_n\}_{n\geq 1}$ be a CIFS. We define the set function $S: H(X) \to H(X)$ by,

$$S(E) = \overline{\cup_{n=1}^{\infty} f_n(E)},$$

where the bar denotes the closure of the corresponding set. The set function S defined is a contraction map on $(H(X),h)$ with contraction ratio $c \leq sup_n c_n$.

There exists a unique set $A \in H(X)$ which is the invariant with respect to $\{f_n\}_{n\geq 1}$, that is

$$A = S(A) = \overline{\cup_{n=1}^{\infty} f_n(A)}$$

This non-empty compact invariant set A is called the attractor of the countable iterated function system $\{f_n\}_{n\geq 1}$.

Let A_k be the attractor of the contraction S_k associated to the partial IFS $\{f_n\}_{n=1}^{k}$, for $k \geq 1$.

Then the set $A = \overline{\bigcup_{n=1}^{\infty} A_k}$ is the attractor of CIFS $\{f_n\}_{n\geq 1}$. The attractor of CIFS $\{f_n\}_{n\geq 1}$ is

$$A = \overline{\bigcup_{n=1}^{\infty} A_k} = lim_n \; A_k$$

the limit being taken in $(H(X), h)$. Hence the attractor of CIFS $\{f_n\}_{n\geq 1}$ is approximated by the attractors of partial IFS $\{f_n\}_{n=1}^{k}, k \geq 1$.

Theorem 5.1: Let (X,d) be a complete metric space and $\{X; f_n\}_{n\geq 1}$ be a $CIFS$ with attractor $A \in H(X)$ such that $A = \overline{\bigcup_{n=1}^{\infty} f_n(A)}$. If there exist a sequence of contractions $\{f_{ij}\}_{i\geq 1}$ which converges to $\{f_j\} \, for \, j = 1, 2, \ldots$. Then $\{X; f_{kn}\}_{n\geq 1}$ is a CIFS having attractor A_k for k=1, 2, . . . and this sequence $\{A_k\}_{k\geq 1}$ in $H(X)$ converges to A in $H(X)$.

Proof. We have a sequence of contractions $\{f_{ij}\}_{i\geq 1}$ on X which converges to $\{f_j\}$ on X for $j = 1, 2, \ldots$. Correspondingly, we have CIFSs $\{X; f_{nk}\}_{n\geq 1}$ for $k = 1, 2, \ldots$. Then there exist a unique set $A_k \in H(X)$ which is invariant with respect to the contractions $\{f_{kn}\}_{n\geq 1}$. ie; $A_k = \overline{\bigcup_{n=1}^{\infty} f_{kn}(A_k)}$, for $k = 1, 2, \ldots$. Thus we have a sequence of sets $\{A_k\}$ in $H(X)$. Now using Theorem 3.4 and Theorem 3.7, the result follows.

Theorem 5.2: Let (X,d) be a complete metric space and $\{X; f_n\}_{n\geq 1}$ be a CIFS with attractor $A \in H(X)$ such that $A = \overline{\bigcup_{n=1}^{\infty} f_n(A)}$. If there exist a sequence of contractions $\{f_{ij}\}_{i\geq 1}$ which converges to $\{f_j\}$ for $j = 1, 2, \ldots$. Then there exist a sequence of set functions $\{S_n\}_{n\geq 1}$ on $H(X)$ which converges to S whose fixed point is A.

Proof. As in the case of above theorem, we get CIFSs $\{X; f_{kn}\}_{n\geq 1}$ for $= 1, 2, \ldots$. By definition, the CIFS $f_{kn}\}_{n\geq 1}$ has an associated set function $S_k(A_k) = \overline{\bigcup_{n=1}^{\infty} f_{kn}(A_k)}$, where $A_k \in H(X)$ is the attractor of the given CIFS. Hence we get a sequence of set functions $\{S_n\}_{n\geq 1}$ on $H(X)$.

The case is now similar to Theorem 3.8 and hence the result follows.

6. CONCLUSION

We have discussed some of the properties of Hausdorff metric topology, and proved some results on the convergence of a sequence of compact subsets of a complete metric space. We have extended our study of the convergence in the case of countable IFS and generalised IFS. The nature of convergence in each case is also discussed.

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