

Research Article**A Least Squares Ratio (LSR) Approach to Fuzzy Linear Regression Analysis****Murat Yazici**

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[Received-03/10/2015, Accepted-17/10/2015, Published-02/11/2015]

ABSTRACT

The Ordinary Least Squares (OLS) approach to Fuzzy Linear Regression Analysis which includes a doubly linear adaptive fuzzy regression model is one of the used methods for fitting a regression model and forecasting. The doubly linear adaptive fuzzy regression model based on two linear models: a core regression model and a spread regression model. It aims to minimize the Euclidean distance between the observed dependent fuzzy values Y_i and the obtained dependent fuzzy values Y_i^* . This paper includes a regression method called Least Squares Ratio (LSR) approach to fuzzy linear regression, and comparison of OLS and LSR according to Mean Absolute Error (MAE). In this study, symmetric triangular fuzzy numbers are used, the observed symmetric triangular fuzzy numbers are shown as $Y_i \equiv (c_i, r_i)$, and the obtained symmetric triangular fuzzy numbers are shown as $Y_i^* \equiv (c_i^*, r_i^*)$. c_i and r_i indicate the observed centers and spreads, and c_i^* and r_i^* indicate the obtained centers and spreads, respectively.

Keywords: Estimation; Least squares ratio (LSR) method; Ordinary least squares (OLS) method; Fuzzy linear regression analysis

1. INTRODUCTION

Vague or fuzzy data and application in several fields, such as psychometry, reliability, marketing, quality control, ballistics, ergonomics, image recognition, artificial intelligence, etc. A typical problem where vague data arise is that of assigning numbers to subjective perceptions or to linguistic variables (such as "enough", "good", "sufficiently", etc.). In fact, there are many cases where observations cannot be known or quantified exactly, and, thus, we can only provide an approximate description of them, or intervals to enclose them. For instance,

"in measuring the influence of character size on the reading ability from a video display terminal the reading ability of the subject, which is essentially the experimental output, depends on his/her eyesight, age, the environment, individual responses, and even how tired is the individual. Some of these factors cannot be described accurately and the best description of these kinds of output is that they are fuzzy outputs" [8] [9].

Fuzzy linear regression was first introduced by Tanaka et al. (1980, 1982). Several

developments followed these papers, such as those studied by Chang and Lee (1994a-c, 1996), Diamond (1990), Diamond and Kloeden (1994), Kacprzyk and Fedrizzi (1992), Kim et al. (1996), Moskowitz and Kim (1993), Peters (1994), Redden and Woodall (1994, 1996), Savic and Pedrycz (1991), Tanaka (1987), Tanaka and Watada (1988), Tanaka and Ishibuchi (1991), Tanaka et al. (1995), Xizhao and Minghu (1992). Also several applications and methods have been proposed: for instance, see Chang et al. (1996), Lee (1996), McCauley and Wang (1997), Kim and Bishu (1998), Wang and Tsaur (2000), Kao and Chyu (2002), Nasrabad and Nasrabi (2004), Bargiela et al. (2007), González-Rodríguez et al. (2009), Lu and Wang (2009), Sener and Karsak (2011), Mashinchi and Orgun (2011), Taheri (2012), Kazemi and Hosseinzadeh (2012), Parvathi et al. (2013) and the others.

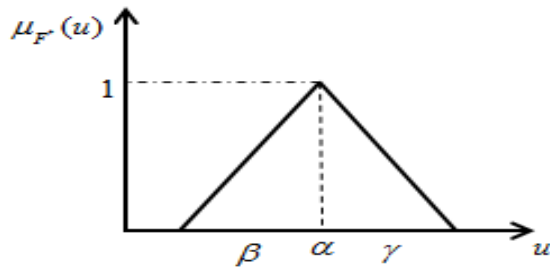


Fig. 1. Triangular membership function. Following Dubois and Prade (1980) and Zimmermann (1985), a fuzzy number may be defined as $F = (\alpha, \beta, \gamma)_{LR}$, where α denotes the center (or mode), β and γ are the left spread (or width) and right spread, respectively, L and R denote the left and right shape functions. When $\beta = \gamma$ and the shape functions are specular (w.r.t. the center), we have a symmetrical fuzzy number, denoted by $F^* = (\alpha, \beta)$. It is common to define a symmetrical fuzzy number by using a triangular membership function also shown in Fig. 1 [8]:

$$\mu_{F^*}(u) = \begin{cases} 1 - \frac{|\alpha - u|}{\beta} & \text{if } \alpha - \beta \leq u \leq \alpha + \beta, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

In the following sections, Least Squares Ratio (LSR) Regression and the OLS approach to fuzzy linear regression analysis developed by D'Urso and Gastaldi are explained. Then, the regression parameters of their doubly linear adaptive fuzzy regression model are estimated by using the LSR method. And, OLS and LSR's performances are compared by Mean Absolute Error (MAE).

2. Least Squares Ratio Regression

The Least Squares Ratio (LSR) method is one of the forecasting techniques in regression analysis. LSR aims to estimate observed values with zero error ($Y = \hat{Y}$, or $Y - \hat{Y} = 0$). It starts with the same goal $Y = \hat{Y}$ as in Ordinary Least Squares. However, it proceeds by dividing through by Y and so $\hat{Y}/Y = 1$ is obtained under an assumption of $Y \neq 0$. Hence, it is obvious that, equations $1 - (\hat{Y}/Y) = 0$ and $(Y - \hat{Y})/Y = 0$ are raised by basic mathematical operations. This final equation is taken into account as the origin of the LSR which minimizes the sum of $[(Y - \hat{Y})/Y]^2$. Consequently the aim of LSR can be written mathematically as follows [7]:

$$\min_{\beta} \sum_{i=1}^n \left(\frac{Y - \hat{Y}}{Y} \right)^2. \quad (2.1)$$

The matrix representation of the regression model is as follows;

$$Y = \beta X + e \quad (2.2)$$

where Y is an $n \times 1$ -vector of observed values; X is an $n \times p$ -vector of the values of dependent variables; n is the number of observations; p is the number of unknown parameters, β is the $p \times 1$ -vector of regression coefficients; e is an $n \times 1$ -vector of error values.

Formula 2.1 can also be written as in formula 2.3, by using Eq. 2.2:

$$\min_{\hat{\beta}_{LSR}} \sum_{i=1}^n \left(\frac{Y - \hat{\beta}_{LSR} X}{Y} \right)^2 \quad (2.3)$$

If $rank(X)$ is equal to p , the formula for estimating β appears as in Eq. 2.4[7]:

$$\hat{\beta}_{LSR} = \left[\left(\frac{X}{Y} \right)' \left(\frac{X}{Y} \right) \right]^{-1} \left(\frac{X}{Y} \right)' Y \quad (2.4)$$

The matrix X/Y is obtained by dividing the values x_{ij} by y_i for $j=1,2,\dots,p$, and X/Y^2 is computed by dividing the values x_{ij} by y_i^2 for $j=1,2,\dots,p$.

3. The OLS Approach to Fuzzy Linear Regression Analysis

The doubly linear adaptive fuzzy regression model defined D'Urso and Gastaldi, 2000 is as follows:

$$\mathbf{c} = \mathbf{c}^* + \varepsilon_c \text{ where } \mathbf{c}^* = \mathbf{X}\mathbf{a}, \quad (3.1)$$

$$\mathbf{r} = \mathbf{r}^* + \varepsilon_r \text{ where } \mathbf{r}^* = \mathbf{c}^*b + \mathbf{1}d, \quad (3.2)$$

$$\delta_i \equiv \delta(Y_i, Y_i^*) = \sqrt{(c_i - c_i^*)^2 + (r_i - r_i^*)^2}. \quad (3.3)$$

The regression parameters are estimated by minimizing the following sum of square errors

$$\varphi(\mathbf{a}, b, d) \equiv \sum_{i=1}^n \delta_i^2 = (\mathbf{c} - \mathbf{c}^*)' (\mathbf{c} - \mathbf{c}^*) + (\mathbf{r} - \mathbf{r}^*)' (\mathbf{r} - \mathbf{r}^*). \quad (3.4)$$

By solving problem (3.4) we have the following iterative least-squares estimates [8]:

$$\mathbf{a} = \frac{1}{(1+b^2)} ((\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'(\mathbf{c} + \mathbf{r}b - \mathbf{1}d)), \quad (3.5)$$

$$b = (\mathbf{a}'\mathbf{X}'\mathbf{X}\mathbf{a})^{-1} (\mathbf{r}'\mathbf{X}\mathbf{a} - \mathbf{a}'\mathbf{X}'\mathbf{1}d), \quad (3.6)$$

$$d = \frac{1}{n} (\mathbf{r}'\mathbf{1} - \mathbf{a}'\mathbf{X}'\mathbf{1}b). \quad (3.7)$$

4. The Proposed LSR Approach to The Doubly Linear Adaptive Fuzzy Regression Model

The Proposed LSR Approach to The Doubly Linear Adaptive Fuzzy Regression Model aims to minimize euclidean distance between the observed dependent fuzzy values Y_i/Y_i and the obtained dependent fuzzy values Y_i^*/Y_i .

The proposed LSR approach is defined as follows;

$$\mathbf{1} = \begin{pmatrix} c^* \\ c \end{pmatrix} + \mathbf{r}\varepsilon_{1c} \text{ where } \begin{pmatrix} c^* \\ c \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ \mathbf{c} \end{pmatrix} \mathbf{a}, \quad (4.1)$$

$$\mathbf{1} = \begin{pmatrix} r^* \\ r \end{pmatrix} + \mathbf{r}\varepsilon_{1r} \text{ where } \begin{pmatrix} r^* \\ r \end{pmatrix} = \begin{pmatrix} c^* \\ c \end{pmatrix} b + \begin{pmatrix} 1 \\ r \end{pmatrix} d, \quad (4.2)$$

where X/c , computed by dividing the values x_{ij} by c_i for $i=1,2,\dots,n$ and $j=1,2,\dots,p$, is a $n \times (k+1)$ -matrix containing the input ratio variables; \mathbf{a} is a column $(k+1)$ -vector containing the regression parameters of the first model called core regression model; c^*/c is the vector of the obtained center ratios, having dimension $n \times 1$; \mathbf{r} is the vector of the observed spread values, having

where \mathbf{X} is a $n \times (k+1)$ -matrix containing the input variables; \mathbf{a} is a column $(k+1)$ -vector containing the regression parameters of the first model called core regression model; \mathbf{c} and \mathbf{c}^* are the vector of the observed centers and the vector of the obtained centers, respectively, both having dimension $n \times 1$; \mathbf{r} and \mathbf{r}^* are the vector of the observed spreads and the vector of the obtained spreads, respectively, both having dimension $n \times 1$; $\mathbf{1}$ is a $n \times 1$ -vector of all 1's, b and d are regression parameters for the second regression model called spread regression model.

The OLS approach to Fuzzy Linear Regression Analysis aims to minimize euclidean distance between the observed dependent fuzzy values Y_i and the obtained dependent fuzzy values Y_i^* . The Euclidean distance between two symmetrical fuzzy numbers $Y_i \equiv (c_i, r_i), Y_i^* \equiv (c_i^*, r_i^*)$ is as follows:

dimension $n \times 1$; r^*/r is the vector of the obtained spread ratios, having dimension $n \times 1$; $1/r$ is computed by dividing the values 1 by r_i for $i = 1, 2, \dots, n$, having dimension $n \times 1$; $\mathbf{1}$ is a $n \times 1$ -vector of all 1's, b and d are regression parameters for the second regression model called spread regression model.

The proposed LSR approach to the doubly linear adaptive fuzzy regression model aims to minimize euclidean distance between the observed ratio values Y_i/Y_i and the obtained ratio values Y_i^*/Y_i . The

Euclidean distance between $\frac{Y_i}{Y_i} \equiv \left(\frac{c_i}{c_i}, \frac{r_i}{r_i}\right) = (1,1)$ and $\frac{Y_i^*}{Y_i} \equiv \left(\frac{c_i^*}{c_i}, \frac{r_i^*}{r_i}\right)$ is as follows:

$$\delta_i \equiv \delta\left(\frac{Y_i}{Y_i}, \frac{Y_i^*}{Y_i}\right) = \delta\left(1, \frac{Y_i^*}{Y_i}\right) = \sqrt{\left(1 - \frac{c_i^*}{c_i}\right)^2 + \left(1 - \frac{r_i^*}{r_i}\right)^2} \quad (4.3)$$

The regression parameters are estimated by minimizing the following sum of square ratios (with matrix notation):

$$\varphi(\mathbf{a}, b, d) \equiv \sum_{i=1}^n \delta_i^2 = \left(\mathbf{1} - \begin{pmatrix} c^* \\ c \end{pmatrix}\right)' \left(\mathbf{1} - \begin{pmatrix} c^* \\ c \end{pmatrix}\right) + \left(\mathbf{1} - \begin{pmatrix} r^* \\ r \end{pmatrix}\right)' \left(\mathbf{1} - \begin{pmatrix} r^* \\ r \end{pmatrix}\right) \quad (4.4)$$

Theorem 1 (least squares ratio iterative solutions). Recursive solution for the problem of the least squares ratio estimation with fuzzy data are given by

$$\mathbf{a} = \frac{1}{1+b^2} \left(\left(\frac{\mathbf{X}}{c}\right)' \left(\frac{\mathbf{X}}{c}\right) \right)^{-1} \left(\left(\frac{\mathbf{X}}{c}\right)' \left(\mathbf{1} + \mathbf{1}b - \left(\frac{1}{r}\right)bd\right) \right), \quad (4.5)$$

$$b = \left(\mathbf{a}' \left(\frac{\mathbf{X}}{c}\right)' \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} \right)^{-1} \left(\mathbf{1}' \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} - \mathbf{a}' \left(\frac{\mathbf{X}}{c}\right)' \left(\frac{1}{r}\right) d \right), \quad (4.6)$$

$$d = \left(\left(\frac{1}{r}\right)' \left(\frac{1}{r}\right) \right)^{-1} \left(\mathbf{1}' \left(\frac{1}{r}\right) - \mathbf{a}' \left(\frac{\mathbf{X}}{c}\right)' \left(\frac{1}{r}\right) b \right). \quad (4.7)$$

$$\begin{aligned} \varphi(\mathbf{a}, b, d) &= \left(\mathbf{1} - \begin{pmatrix} c^* \\ c \end{pmatrix}\right)' \left(\mathbf{1} - \begin{pmatrix} c^* \\ c \end{pmatrix}\right) + \left(\mathbf{1} - \begin{pmatrix} r^* \\ r \end{pmatrix}\right)' \left(\mathbf{1} - \begin{pmatrix} r^* \\ r \end{pmatrix}\right) \\ &= \left(\mathbf{1} - \begin{pmatrix} \mathbf{X} \\ c \end{pmatrix} \mathbf{a}\right)' \left(\mathbf{1} - \begin{pmatrix} \mathbf{X} \\ c \end{pmatrix} \mathbf{a}\right) + \left(\mathbf{1} - \left(\begin{pmatrix} \mathbf{X} \\ c \end{pmatrix} \mathbf{a} b + \begin{pmatrix} 1 \\ r \end{pmatrix} d\right)\right)' \left(\mathbf{1} - \left(\begin{pmatrix} \mathbf{X} \\ c \end{pmatrix} \mathbf{a} b + \begin{pmatrix} 1 \\ r \end{pmatrix} d\right)\right) \\ &= n - 2\mathbf{1}' \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} + \left(\frac{\mathbf{X}}{c}\right)' \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} + n - 2\mathbf{1}' \left(\begin{pmatrix} \mathbf{X} \\ c \end{pmatrix} \mathbf{a} b + \begin{pmatrix} 1 \\ r \end{pmatrix} d\right) \\ &\quad + \left(\begin{pmatrix} \mathbf{X} \\ c \end{pmatrix} \mathbf{a} b + \begin{pmatrix} 1 \\ r \end{pmatrix} d\right)' \left(\begin{pmatrix} \mathbf{X} \\ c \end{pmatrix} \mathbf{a} b + \begin{pmatrix} 1 \\ r \end{pmatrix} d\right) \\ &= 2n - 2\mathbf{1}' \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} + \mathbf{a}' \left(\frac{\mathbf{X}}{c}\right)' \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} - 2\mathbf{1}' \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} b - 2\mathbf{1}' \left(\frac{1}{r}\right) d \\ &\quad + \mathbf{a}' \left(\frac{\mathbf{X}}{c}\right)' \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} b^2 + 2\mathbf{a}' \left(\frac{\mathbf{X}}{c}\right)' \left(\frac{1}{r}\right) b d + \left(\frac{1}{r}\right)' \left(\frac{1}{r}\right) d^2 \quad (4.8) \end{aligned}$$

Let $\mathbf{0}$ be a zero column $(k+1)$ -vector. We assume throughout the paper that the matrix $\mathbf{X}'\mathbf{X}$ is non-singular. By partially differentiating and equating to a zero vector we have following equations;

$$\frac{\partial}{\partial \mathbf{a}} \varphi(\mathbf{a}, b, d) = \mathbf{0}$$

$$\Rightarrow -\left(\frac{X}{c}\right)' \mathbf{1} + \left(\frac{X}{c}\right)' \left(\frac{X}{c}\right)' \mathbf{a} - \left(\frac{X}{c}\right)' \mathbf{1} b + \left(\frac{X}{c}\right)' \left(\frac{X}{c}\right)' \mathbf{a} b^2 + \left(\frac{X}{c}\right)' \left(\frac{1}{r}\right) b d = \mathbf{0}, (4.9)$$

$$\frac{\partial}{\partial b} \varphi(\mathbf{a}, b, d) = 0 \quad ,$$

$$\Rightarrow -\mathbf{1}' \left(\frac{X}{c}\right)' \mathbf{a} + \mathbf{a}' \left(\frac{X}{c}\right)' \left(\frac{X}{c}\right)' \mathbf{a} b + \mathbf{a}' \left(\frac{X}{c}\right)' \left(\frac{1}{r}\right) d = 0 (4.10)$$

$$\frac{\partial}{\partial d} \varphi(\mathbf{a}, b, d) = 0 \quad .$$

$$\Rightarrow -\mathbf{1}' \left(\frac{1}{r}\right)' + \mathbf{a}' \left(\frac{X}{c}\right)' \left(\frac{1}{r}\right)' b + \left(\frac{1}{r}\right)' \left(\frac{1}{r}\right)' d = 0 . (4.11)$$

Proposition 1. The sum (and; hence; the mean) of the n errors $r(\varepsilon_{1c})/r = (\varepsilon_{1c})$ is zero, i.e.

$$\left(\frac{1}{r}\right)' \left(\mathbf{1} - \left(\frac{c^*}{c}\right)\right) = 0 \quad (4.12)$$

Proof of Proposition 1. Let, $\mathbf{0}_n$ be a zero column n-vector (and $\mathbf{0}$ a zero columnvector having

dimension $k+1$). By (4.9) and the relation $\left(\frac{c^*}{c}\right) = \left(\frac{X}{c}\right)' \mathbf{a}$, we have

$$\Rightarrow -\left(\frac{X}{c}\right)' \mathbf{1} + \left(\frac{X}{c}\right)' \left(\frac{X}{c}\right)' \mathbf{a} - \left(\frac{X}{c}\right)' \mathbf{1} b + \left(\frac{X}{c}\right)' \left(\frac{X}{c}\right)' \mathbf{a} b^2 + \left(\frac{X}{c}\right)' \left(\frac{1}{r}\right) b d = \mathbf{0}$$

$$\Rightarrow \left(\frac{X}{c}\right)' \left(-\mathbf{1} + \left(\frac{c^*}{c}\right) - \mathbf{1} b + \left(\frac{c^*}{c}\right) b^2 + \left(\frac{1}{r}\right) b d\right) = \mathbf{0}$$

(note here the zero vector has dimension $k+1$)

$$\Rightarrow \left(\frac{X}{c}\right)' \left(\frac{X}{c}\right)' \left(-\mathbf{1} + \left(\frac{c^*}{c}\right) - \mathbf{1} b + \left(\frac{c^*}{c}\right) b^2 + \left(\frac{1}{r}\right) b d\right) = \left(\frac{X}{c}\right)' \left(\frac{X}{c}\right)' \mathbf{0}_n$$

(note here the zero vector has dimension n),

by multiplying to the left by $\left[\left(\frac{X}{c}\right)' \left(\frac{X}{c}\right)'\right]^{-1}$, we get

$$\Rightarrow -\mathbf{1} + \left(\frac{c^*}{c}\right) - \mathbf{1} b + \left(\frac{c^*}{c}\right) b^2 + \left(\frac{1}{r}\right) b d = \mathbf{0}_n$$

$$\Rightarrow \left(\frac{1}{r}\right)' \left(-\mathbf{1} + \left(\frac{c^*}{c}\right) - \mathbf{1} b + \left(\frac{c^*}{c}\right) b^2 + \left(\frac{1}{r}\right) b d\right) = \left(\frac{1}{r}\right)' \mathbf{0}_n$$

$$\Rightarrow \left(\frac{1}{r}\right)' \left(\mathbf{1} - \left(\frac{c^*}{c}\right)\right) = -\left(\frac{1}{r}\right)' \mathbf{1} b + \left(\frac{1}{r}\right)' \left(\frac{c^*}{c}\right) b^2 + \left(\frac{1}{r}\right)' \left(\frac{1}{r}\right) b d ,$$

by noticing that $\left(\frac{c^*}{c}\right) = \left(\frac{X}{c}\right)' \mathbf{a}$, we have

$$\Rightarrow \left(\frac{1}{r}\right)' \left(\mathbf{1} - \left(\frac{c^*}{c}\right)\right) = -\left(\frac{1}{r}\right)' \mathbf{1} b + \left(\frac{1}{r}\right)' \left(\frac{X}{c}\right)' \mathbf{a} b^2 + \left(\frac{1}{r}\right)' \left(\frac{1}{r}\right) b d$$

Finally, we can note that

$$\Rightarrow -\mathbf{1}'\left(\frac{\mathbf{1}}{r}\right) + \mathbf{a}'\left(\frac{\mathbf{X}}{c}\right)\left(\frac{\mathbf{1}}{r}\right)b + \left(\frac{\mathbf{1}}{r}\right)\left(\frac{\mathbf{1}}{r}\right)d = 0$$

the above relationship is equivalent to (4.7). □

Proposition 2. The sum (and, hence, the mean) of the n errors $r(\varepsilon_{1r})/r = (\varepsilon_{1r})$ is zero; i.e.

$$\left(\frac{\mathbf{1}}{r}\right)' \left(\mathbf{1} - \left(\frac{r^*}{r}\right)\right) = 0 \quad (4.13)$$

Proof of Proposition 2. By (4.11), we have

$$\Rightarrow -\mathbf{1}'\left(\frac{\mathbf{1}}{r}\right) + \mathbf{a}'\left(\frac{\mathbf{X}}{c}\right)\left(\frac{\mathbf{1}}{r}\right)b + \left(\frac{\mathbf{1}}{r}\right)\left(\frac{\mathbf{1}}{r}\right)d = 0$$

$$\Rightarrow -\left(\frac{\mathbf{1}}{r}\right)' \mathbf{1} + \left(\frac{\mathbf{1}}{r}\right)' \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} b + \left(\frac{\mathbf{1}}{r}\right)' \left(\frac{\mathbf{1}}{r}\right) d = 0$$

$$\Rightarrow \left(\frac{\mathbf{1}}{r}\right)' \left(\mathbf{1} - \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} b - \left(\frac{\mathbf{1}}{r}\right) d\right) = 0$$

By $\left(\frac{r^*}{r}\right) = \left(\frac{\mathbf{X}}{c}\right) \mathbf{a} b + \left(\frac{\mathbf{1}}{r}\right) d$, we have our thesis. □

Remark 1. Propositions 1 and 2 imply the equality of the following averages:

$$E\left(\frac{\mathbf{1}}{r}\right) = E\left(\frac{c^*/c}{r}\right) \text{ and } E\left(\frac{\mathbf{1}}{r}\right) = E\left(\frac{r^*/r}{r}\right). \quad (4.14)$$

$$\text{so, } E(\varepsilon_{1c}) = E\left(\frac{\mathbf{1}}{r}\right) - E\left(\frac{c^*/c}{r}\right) = 0, \quad (4.15)$$

$$E(\varepsilon_{1r}) = E\left(\frac{\mathbf{1}}{r}\right) - E\left(\frac{r^*/r}{r}\right) = 0. \quad (4.16)$$

Because \mathbf{r} is a non-random $n \times 1$ matrix,

$$E(r\varepsilon_{1c}) = rE(\varepsilon_{1c}) = r \cdot 0 = 0, \quad (4.17)$$

$$E(r\varepsilon_{1r}) = rE(\varepsilon_{1r}) = r \cdot 0 = 0. \quad (4.18)$$

4. Applicative examples

In this section, two examples with one and several independent variables are shown to illustrate the proposed approach, respectively.

Table 1. Input (crisp)-output (fuzzy) data

Objects	Visual angle	Subject preference
i	x_i	$Y_i = (c_i, r_i)$
1	14.8	(67.7, 42.7)
2	18.0	(69.8, 32.3)
3	22.9	(89.6, 27.1)
4	31.5	(88.5, 26.0)
5	50.3	(71.9, 34.4)
6	126.0	(30.2, 44.8)

Table 2. Chang et al.'s data results

c/c	c*/c	r/r	r*/r	c	c*		r	r*		c-c*		r-r*		
					ols	lsr		ols	lsr	ols	lsr	ols	lsr	
1	1.16	1	1.02	67.7	82.14	78.84	42.7	31.52	43.54	14.44	11.14	11.18	0.84	
1	1.11	1	1.09	69.8	80.76	77.50	32.3	31.86	35.09	10.96	7.70	0.44	2.79	
1	0.84	1	1.00	89.6	78.65	75.44	27.1	32.37	27.14	10.95	14.16	5.27	0.04	
1	0.81	1	1.00	88.5	74.96	71.84	26.0	33.26	26.10	13.54	16.66	7.26	0.10	
1	0.89	1	0.92	71.9	66.87	63.95	34.4	35.21	31.88	5.03	7.95	0.81	2.52	
1	1.07	1	0.94	30.2	34.32	32.18	44.8	43.08	42.33	4.12	1.98	1.72	2.47	
										mae =	9.84	9.93	4.45	1.46

$$\frac{1}{6} \sum_{i=1}^6 (c/c - c^*/c) \cong 0 \quad \frac{1}{6} \sum_{i=1}^6 (r/r - r^*/r) \cong 0$$

Iteration number: 318

Example 1.(one independent variable). The data in Table 1 were drawn by Changet al. (1996). These data refer to the effects of the position of a video displayterminal on an operator. The dependent fuzzy variable is the subject preferencegiven by a crisp value, in a scale 0-100, and a corresponding spread, while the(crisp) independent variable is the visual angle of the display [8].

The parameters of the doubly linear adaptive fuzzy regression model have been estimated as follows by using the proposed LSR approach;

$$\mathbf{1} = \begin{pmatrix} c^* \\ c \end{pmatrix} + \mathbf{r}\varepsilon_{1c}, \quad \begin{pmatrix} c^* \\ c \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ c \end{pmatrix} \mathbf{a}, \text{core regression model,}$$

$$\mathbf{1} = \begin{pmatrix} r^* \\ r \end{pmatrix} + \mathbf{r}\varepsilon_{1r}, \quad \begin{pmatrix} r^* \\ r \end{pmatrix} = \begin{pmatrix} c^* \\ c \end{pmatrix} b + \begin{pmatrix} 1 \\ r \end{pmatrix} d, \text{spread regression model,}$$

and the parameters were found as follows;

$$\mathbf{a} = (85.05, -0.42)',$$

$$b = 0.61,$$

$$d = 13.24.$$

Table 2 indicates Chang et al.'s data results. According to mae, we can say that the OLS Approach gave a little bit better results than the LSR Approach while estimating center values c_i . Also, we can say that the LSR Approach showed better performance than the OLS Approach while estimating spread values r_i .

Table 3. Input (crisp)-output (fuzzy) data

Objects <i>i</i>	Crisp input			Fuzzy output $Y_i = (c_i, r_i)$
	x_{1i}	x_{2i}	x_{3i}	
1	3	5	9	(96, 42)
2	14	8	3	(120, 47)
3	7	1	4	(52, 33)
4	11	7	3	(106, 45)
5	7	12	15	(189, 79)
6	8	15	10	(194, 65)
7	3	9	6	(107, 42)
8	12	15	11	(216, 78)
9	10	5	8	(108, 52)
10	9	7	4	(103, 44)

One can note that, as expected, the operator's preference rapidly decades, as the visual angle increases[8].

Example 2. (*several independent variables*). The procedure was applied to the datashown in Table 3, drawn by Tanaka (1987).

The parameters were calculated as follows;

$$\mathbf{a} = (1.78, 3.11, 7.66, 5.01)',$$

$$b = 1.02,$$

$$d = -1.14.$$

and the results are summarized below Table 4.

Table 4. Tanaka’s data results

c/c	c*/c	r/r	r*/r	c	c*		r	r*		c-c*		r-r*	
					ols	lsr		ols	lsr	ols	lsr	ols	lsr
1	0.98	1	0.98	96	93.86	94.49	42	42.38	41.04	2.14	1.51	0.38	0.96
1	1.01	1	1.00	120	122.04	121.58	47	50.63	47.44	2.04	1.58	3.63	0.44
1	0.98	1	1.00	52	50.11	51.23	33	29.56	33.12	1.89	0.77	3.44	0.12
1	0.99	1	0.98	106	104.13	104.60	45	45.38	44.16	1.87	1.40	0.38	0.84
1	1.00	1	1.01	189	193.31	190.60	79	71.51	80.15	4.31	1.60	7.49	1.15
1	0.99	1	0.99	194	192.58	191.62	65	71.30	64.37	1.42	2.38	6.30	0.63
1	1.03	1	1.02	107	109.12	110.08	42	46.55	42.95	2.12	3.08	4.55	0.95
1	0.97	1	0.97	216	211.71	209.06	78.0	76.90	75.89	4.29	6.94	1.10	2.11
1	1.03	1	1.03	108	112.48	111.23	52	47.86	53.51	4.48	3.23	4.14	1.51
1	1.00	1	1.00	103	102.67	103.39	44	44.96	43.93	0.33	0.39	0.96	0.07

$$\frac{1}{6} \sum_{i=1}^6 (c/c - c^*/c) \cong 0 \quad \frac{1}{6} \sum_{i=1}^6 (r/r - r^*/r) \cong 0$$

$$mae = 2.49 \quad 2.29 \quad 3.24 \quad 0.88$$

Iteration number: 366

Table 4 indicates Tanaka.’s data results. According to mae, we can say that the LSR Approach gave better results than the OLS Approach while estimating center values c_i and spread values r_i .

5.CONCLUSIONS AND FUTURE WORK

In this study,a regression method calledLeast Squares Ratio (LSR) approach to fuzzy linear regression was explained as a alternative method to other techniques.

According to Mean Absolute Error (MAE), we can say that the LSR Approach generally gave better results than the OLS Approach. In addition to this, in case of the presence of outliers and/or extrem values, the LSR Approach will show better performance than OLS because the LSR Approach is more sensitive to them.

For future work, the LSR and OLS Approaches can be compared about which technique give better results by using different error criteria. Also, their performance can be explored in case of different fuzzy numbers such as non-simetric triangular and trapezoidal fuzzy numbers etc.

ACKNOWLEDGEMENTS

I would like to thank Jforce Information Technologies Inc. Istanbul, Turkey for supporting this study. I would also like to thank Dr. Oguz Akbilgic for sharing his knowledge with me.

Appendix A.

A.1. R programing codes for Chang et al.’s data.

```

/*Input data*/

c<-matrix(c(67.7, 69.8, 89.6, 88.5, 71.9, 30.2))
*/observed centers ofthe dependent fuzzy variable;
r<-matrix(c(42.7, 32.3, 27.1, 26, 34.4, 44.8))
*/observed spreads of the dependent fuzzy variable;
x1<-matrix(c(14.8, 18, 22.9, 31.5, 50.3, 126)) */matrix
of observed independent crisp variables;

ObjNum<-nrow(x1) */number of
statistical objects;
    
```



```

IndVarNum<-ncol(x1)                */number of
independent variables;
One<-matrix(numeric(ObjNum)+1)     */column vector
of all 1's;                          /* Output */
X<-cbind(One,x1)                   */data regression
matrix;                               print(a)
Z<-as.matrix(1:ObjNum)             print(b)
Z<-cbind(Z,Z)                      print(d)
s<-as.matrix(1:ObjNum)             print(OneStar)
for(i in 1:ObjNum){                print(OnerStar)
  s[i,1]<-1/r[i,1]                  */calculating of
matrix (1/r);                       print(cStar)
  for(j in 1:(IndVarNum+1)){        print(rStar)
    Z[i,j]<-X[i,j]/c[i,1]           */calculating of data
ratio regression matrix;            print(IterNum)
  }
}

N<-t(Z)%*%Z
M<-solve(N)

/* Doubly linear adaptive fuzzy regression model: iterative
solution */

threshold<-1e-8
IterNum<-0
a <-matrix(numeric(2)+0.0001)      */arbitrary starting
value;
b <- 0.0001                          */arbitrary starting value;
d <- 0.0001                          */arbitrary starting value;
aOld <- 0*/startinng value, 0;
bOld <- 0*/startinng value, 0;
dOld <- 0 */startinng value, 0;
while (abs(a-aOld)||abs(as.numeric(b)-
bOld)||abs(as.numeric(d)-dOld)>threshold){
  IterNum<-IterNum+1
  aOld <- a
  bOld <- as.numeric(b)
  dOld <- as.numeric(d)
  a<-(M%*%t(Z)%*%(One+One*as.numeric(b)-
s*as.numeric(b)*as.numeric(d)))/(1+as.numeric(b)^2)
  b<-
solve(t(a)%*%t(Z)%*%Z%*%a)%*%(t(One)%*%Z%*%
a-t(a)%*%t(Z)%*%s*as.numeric(d))
  d<-solve(t(s)%*%s)%*%(t(One)%*%s-
t(a)%*%t(Z)%*%s*as.numeric(b))
}
OneStar<-Z%*%a */calculating of ratio c* matrix;
cStar<-as.matrix(1: ObjNum)
for(i in 1:ObjNum){
  cStar[i,1]<-OneStar[i,1]*c[i,1] */calculating c* matrix;
}
OnerStar<-OneStar*as.numeric(b)+s*as.numeric(d)
*/calculating of ratio r* matrix;
rStar<-as.matrix(1: ObjNum)
for(i in 1:ObjNum){
  rStar[i,1]<-OnerStar[i,1]*r[i,1] */calculating of r*
matrix;
}

```

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