International Journal of Chemical Sciences and Applications ISSN 0976-2590.
Vol 2, Issue 2, June-2011, pp 123-133 http://www.bipublication.com

EFFECTS OF MAGNETIC FIELD ON UNSTEADY HEAT AND MASS TRANSFER FLOW OF A VISCOUS CONDUCTING FLUID PAST AN INFINITE POROUS PLATE WITH TIME DEPENDENT SUCTION AND HEAT FLUX

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ABSTRACT

This chapter presents a theoretical analysis of unsteady MHD free convective and mass transfer flow of viscous incompressible electrically conducting fluid past an infinite porous vertical plate in non homogeneous porous medium taking time dependent suction velocity at the plate which decreases exponentially with time and a constant heat flux into account. We consider a Cartesian coordinate system (x, y, z) rotating uniformly with the liquid in a rigid state of rotation with a constant angular velocity (0, 0, Ω) about the z-axis is considered. The vertical plate is assumed in the plane z=0 and z-axis is taken normal to the plate pointing towards the flow medium. The approximate solutions are obtained for velocity, temperature and mass concentration fields. The effects of Grashof number for heat transfer, Grashof number for mass transfer, Prandtl number, Magnetic parameter, Schmidt number and rotational parameter on velocity, temperature and mass concentration are discussed through graphs.
Also the expressions for skin-friction and rate of heat and mass transfer coefficients are evaluated and discussed numerically through tables.

INTRODUCTION

Many transport processes exist in nature and in industrial applications in which the transfer of heat and mass occurs simultaneously as a result of combined buoyancy effects of thermal diffusion and diffusion of chemical species. In the last four decades extensive research efforts have been devoted to problems on heat and mass transfer in view of their application to astrophysics, geophysics and engineering. The phenomenon of heat and mass transfer is also encountered in chemical processing industries such as food processing and polymer products.

Stokes [1], in his celebrated memoir on the motion of pendulums, first investigated the flow of an incompressible fluid near an infinite flat plate, which is impulsively started from rest into motion in its own plate with a constant velocity. Cole [3] has introduced perturbation methods in applied mathematics. Several authors including Bejan & Khair [4], Huang & Chen [5], Jang & Ni [6], Adnan [7], Hunt & Tien [8], Jang et al.[9], Bejan [10], Gholami & Singh [11], Ganapathy [12], Angirasa et al.[13], [14], Inaba et al.[18], Sonth et al.[19], Chamkha et al, [20], Singh & Singh [21] have studied problems on heat and mass transfer of viscous incompressible fluids under different physical situations.

Recently, Singh [22] has presented a detailed analysis a problem on unsteady free convection and mass transfer flow of an incompressible viscous liquid through a porous medium past an infinite vertical porous plate subject to time dependent suction velocity normal to the plate using perturbation technique. Singh et al. [17] and Singh [15, 16] studied the suction velocity distribution on viscous flow and heat transfer problems along flat and vertical porous plates. Singh et al. [23] have studied the Heat and mass transfer flow of a viscous fluid past an infinite porous plate with time dependent suction and heat flux.
In the above stated investigations, the suction velocity is considered either constant or periodic. However in engineering problems there are situations where the use of suction velocity exponentially decreasing with time is inevitable. This chapter presents a theoretical analysis of unsteady MHD free convective and mass transfer flow of viscous incompressible electrically conducting fluid past an infinite porous vertical plate in non homogeneous porous medium taking time dependent suction velocity at the plate which decreases exponentially with time and a constant heat flux into account.

FORMULATION OF THE PROBLEM

Unsteady laminar free convection and mass transfer flow of a viscous incompressible and electrically conducting fluid past an infinite vertical porous plate in non homogeneous porous medium is considered. A uniform transverse magnetic field is applied perpendicular to the plate.
The Cartesian coordinate system $\left(x,y,z\right)$ rotating uniformly with the liquid in a rigid state of rotation with a constant angular velocity $\left(0,0,\Omega\right)$ about z-axis. The vertical plate is assumed to coincide with the plane $z=0$ and z-axis is taken normal to the plate pointing towards the flow medium.

The governing equations are as follows

$$\frac{\partial u}{\partial t}-2\Omega v-w_0\left(1+\varepsilon e^{-nt}\right)\frac{\partial u}{\partial z}=g\beta^*(T-T_\infty)+g\beta(C-C_\infty)+\upsilon\frac{\partial^2 u}{\partial z^2}-\frac{\upsilon}{k}u-\frac{\sigma_e B_0^2 u}{\rho} \qquad (1)$$

$$\frac{\partial v}{\partial t}+2\Omega u-w_0\left(1+\varepsilon e^{-nt}\right)\frac{\partial v}{\partial z}=\upsilon\frac{\partial^2 v}{\partial z^2}-\frac{\upsilon}{k}v-\frac{\sigma_e B_0^2 v}{\rho} \qquad (2)$$

$$\frac{\partial T}{\partial t}-w_0\left(1+\varepsilon e^{-nt}\right)\frac{\partial T}{\partial z}=\frac{K^1}{\rho c_p}\frac{\partial^2 T}{\partial z^2} \qquad (3)$$

$$\frac{\partial C}{\partial t}-w_0\left(1+\varepsilon e^{-nt}\right)\frac{\partial C}{\partial z}=D\frac{\partial^2 C}{\partial z^2} \qquad (4)$$

The corresponding boundary conditions are

$$t>0:\; u=v=w_0\left(1+\varepsilon e^{-nt}\right),\quad \frac{\partial T}{\partial z}=\frac{-q}{K^1},$$

$$C = C_w + \varepsilon\left(C_w - C_\infty\right)e^{-nt} \qquad \text{at} \quad z = 0$$
$$u = v \to 0, \quad T \to T_\infty, \quad C \to C_\infty \qquad \text{as} \quad z \to \infty \tag{5}$$

NON-DIMENSIONALISATION

Now we introduce the following non-dimensional variables as follows:

$$u^* = \frac{u}{w_0}, \quad v^* = \frac{v}{w_0}, \quad z^* = \frac{w_0^2 z}{\upsilon}, \quad n^* = \frac{\upsilon n}{w_0^2}, \quad t^* = \frac{w_0^2 t}{\upsilon},$$
$$T^* = \frac{(T - T_\infty)K^1}{(q\upsilon)}, \quad C^* = \frac{(C - C_\infty)}{(C_w - C_\infty)}, \quad k^* = \frac{\upsilon k}{w_0^2}, \quad K^* = \frac{w_0^2 K^1}{\upsilon}$$

Using the above variables, the equations (1) to (4) (neglecting the stars over them) are reduced to

$$\frac{\partial u}{\partial t} - 2Ev - \left(1 + \varepsilon e^{-nt}\right)\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial z^2} + GrT + GmC - \frac{u}{k} - Mu \tag{6}$$
$$\frac{\partial v}{\partial t} + 2Eu - \left(1 + \varepsilon e^{-nt}\right)\frac{\partial v}{\partial z} = \frac{\partial^2 v}{\partial z^2} - \frac{v}{k} - Mv \tag{7}$$
$$\frac{\partial T}{\partial t} - \left(1 + \varepsilon e^{-nt}\right)\frac{\partial T}{\partial z} = \frac{1}{\Pr}\frac{\partial^2 T}{\partial z^2} \tag{8}$$
$$\frac{\partial C}{\partial t} - \left(1 + \varepsilon e^{-nt}\right)\frac{\partial C}{\partial z} = \frac{1}{Sc}\frac{\partial^2 C}{\partial z^2} \tag{9}$$

Where

$\Pr = \frac{\mu c_p}{K}$ (Prandtl number), $E = \frac{\Omega \upsilon}{w_0^2}$ (Rotational parameter),

$Sc = \frac{\upsilon}{D}$ (Schmidt number), $Gm = \frac{g\beta\upsilon\left(C_w - C_\infty\right)}{w_0^3}$ (Modified Grashof number),

$Gr = \frac{g\beta^*\upsilon^2 q}{K^1 w_0^4}$ (Grashof number) and $M = \frac{\sigma_e B_0^2 \upsilon}{\rho w_0^2}$ (Magnetic parameter).

Assuming $W = u + iv$, the equations (6) and (7) give

$$\frac{\partial W}{\partial t} + 2iEW - \left(1 + \varepsilon e^{-nt}\right)\frac{\partial W}{\partial z} = \frac{\partial^2 W}{\partial z^2} + GrT + GmC - \frac{W}{k} - MW \tag{10}$$

The boundary conditions (5) become

$$t > 0: \quad W = \left(1 + \varepsilon e^{-nt}\right), \quad \frac{dT}{dz} = -1, \quad C = \left(1 + \varepsilon e^{-nt}\right) \qquad \text{at} \quad z = 0$$

(11)

$$W \to 0, \quad T \to 0, \quad C \to 0 \qquad \text{as} \quad z \to \infty$$

SOLUTION OF THE PROBLEM

To obtain the solution of (10), (9) and (8), the following Lighthill [2] we assume

$$W = W_1(z) + \varepsilon W_2(z) e^{-nt} + ...$$
$$T = T_1(z) + \varepsilon T_2(z) e^{-nt} + ... \qquad (12)$$
$$C = C_1(z) + \varepsilon C_2(z) e^{-nt} + ...$$

Using (12) in equations (8), (9) and (10) and comparing the coefficients of harmonic and non-harmonic terms, we obtain the following six equations

$$T_1^{11}(z) + \Pr T_1^1(z) = 0 \qquad (13)$$
$$T_2^{11}(z) + \Pr T_2^1(z) + n \Pr T_2(z) = -\Pr T_1^1(z) \qquad (14)$$
$$C_1^{11}(z) + ScC_1^1(z) = 0 \qquad (15)$$
$$C_2^{11}(z) + ScC_2^1(z) + nScC_2(z) = -ScC_1^1(z) \qquad (16)$$
$$W_1^{11}(z) + W_1^1(z) - \left(2iE + \frac{1}{k} + M\right) W_1(z) = -GrT_1(z) - GmC_1(z) \qquad (17)$$
$$W_2^{11}(z) + W_2^1(z) - \left(2iE + \frac{1}{k} + M - n\right) W_2(z) = -GrT_2(z) - GmC_2(z) - W_1^1(z) \qquad (18)$$

Substituting (12) in boundary conditions (11) we obtain

$$W_1 = 1 = W_2, \quad C_1 = 1 = C_2, \quad \frac{dT_1}{dz} = -1, \quad \frac{dT_2}{dz} = 0 \quad \text{at} \quad z = 0$$
$$(19)$$
$$T_1 = T_2 \to 0, \quad C_1 = C_2 \to 0 \qquad \text{as} \quad z \to \infty$$

The solution of (13) to (18) under the transformed boundary conditions (19) yield

$$T_1(z) = \frac{1}{\Pr} e^{-\Pr z} \qquad (20)$$
$$T_2(z) = -\frac{\Pr}{nm_1} e^{-m_1 z} + \frac{e^{-\Pr z}}{n} \qquad (21)$$
$$C_1(z) = e^{-Scz} \qquad (22)$$
$$C_2(z) = \left(1 - \frac{Sc}{n}\right) e^{-m_2 z} + \frac{Sc}{n} e^{-Scz} \qquad (23)$$
$$u_1(z) = -P_1 e^{-\Pr z} - P_2 e^{-Scz} + e^{-A_1 z}\left(A_{13}\cos B_1 z + A_{14}\sin B_1 z\right) \qquad (24)$$
$$v_1(z) = -Q_1 e^{-\Pr z} - Q_2 e^{-Scz} + e^{-A_1 z}\left(A_{14}\cos B_1 z - A_{13}\sin B_1 z\right) \qquad (25)$$
$$u_2(z) = P_6 e^{-m_1 z} - P_7 e^{-m_2 z} - e^{-A_1 z}\left(P_8\cos B_1 z + Q_8 \sin B_1 z\right)$$
$$-P_9 e^{-\Pr z} - P_{10} e^{-Scz} + e^{-A_2 z}\left(P_{11}\cos B_2 z + Q_{11}\sin B_2 z\right) \qquad (26)$$
$$v_2(z) = Q_6 e^{-m_1 z} - Q_7 e^{-m_2 z} - e^{-A_1 z}\left(Q_8\cos B_1 z + P_8 \sin B_1 z\right)$$
$$-Q_9 e^{-\Pr z} - Q_{10} e^{-Scz} + e^{-A_2 z}\left(Q_{11}\cos B_2 z - P_{11}\sin B_2 z\right) \qquad (27)$$

The various constants used above are given in the Appendix.
On putting the values of $T_1(z)$, $T_2(z)$, $C_1(z)$, $C_2(z)$, $W_1(z)$ and $W_2(z)$ in equation (12), we obtain

$$T(z,t) = \frac{1}{\Pr} e^{-\Pr z} + \varepsilon\left(-\frac{\Pr}{nm_1} e^{-m_1 z} + \frac{e^{-\Pr z}}{n}\right) e^{-nt} \tag{28}$$

$$C(z,t) = e^{-Scz} + \varepsilon\left[\left(1 - \frac{Sc}{n}\right) e^{-m_2 z} + \frac{Sc}{n} e^{-Scz}\right] e^{-nt} \tag{29}$$

$$\begin{aligned} u(z,t) = & -P_1 e^{-\Pr z} - P_2 e^{-Scz} + e^{-A_1 z}\left(A_{13}\cos B_1 z + A_{14}\sin B_1 z\right) \\ & +\varepsilon\Big[P_6 e^{-m_1 z} - P_7 e^{-m_2 z} - e^{-A_1 z}\left(P_8 \cos B_1 z + Q_8 \sin B_1 z\right) \\ & -P_9 e^{-\Pr z} - P_{10} e^{-Scz} + e^{-A_2 z}\left(P_{11}\cos B_2 z + Q_{11}\sin B_2 z\right)\Big] e^{-nt} \end{aligned} \tag{30}$$

$$\begin{aligned} v(z,t) = & -Q_1 e^{-\Pr z} - Q_2 e^{-Scz} + e^{-A_1 z}\left(A_{14}\cos B_1 z - A_{13}\sin B_1 z\right) \\ & +\varepsilon\Big[Q_6 e^{-m_1 z} - Q_7 e^{-m_2 z} - e^{-A_1 z}\left(Q_8 \cos B_1 z + P_8 \sin B_1 z\right) \\ & -Q_9 e^{-\Pr z} - Q_{10} e^{-Scz} + e^{-A_2 z}\left(Q_{11}\cos B_2 z - P_{11}\sin B_2 z\right)\Big] e^{-nt} \end{aligned} \tag{31}$$

Knowing the velocity, temperature and concentration fields in the boundary layer, we can calculate the skin-friction, Nusselt number and Sherwood number coefficients respectively at the plate are given by

Skin-friction at the plate: $\tau = \left(\frac{\partial W}{\partial z}\right)_{z=0} = \left(\frac{\partial u}{\partial z}\right)_{z=0} + i\left(\frac{\partial v}{\partial z}\right)_{z=0} = \tau_p + i\tau_s$

$$\tau_p = A_3 + \varepsilon A_4 e^{-nt} \tag{32}$$

$$\tau_s = B_3 + \varepsilon B_4 e^{-nt} \tag{33}$$

Where

$$A_3 = P_1 \Pr + P_2 Sc + B_1 A_{14} - A_1 A_{13}$$

$$A_4 = -m_1 P_6 + m_2 P_7 - B_1 Q_8 + A_1 P_8 + P_9 \Pr + P_{10} Sc + B_2 Q_{11} - A_2 P_{11}$$

$$B_3 = Q_1 \Pr + Q_2 Sc - B_1 A_{13} - A_1 A_{14}$$

$$B_4 = -m_1 Q_6 + m_2 Q_7 - B_1 Q_8 + A_1 P_8 + Q_9 \Pr + Q_{10} Sc - B_2 P_{11} - A_2 Q_{11}$$

Nusselt number: $$Nu = \left(\frac{\partial T}{\partial z}\right)_{z=0} = 1 \tag{34}$$

Sherwood number: $$Sh = \left(\frac{\partial C}{\partial z}\right)_{z=0} = -Sc - \varepsilon\left[m_2\left(1 - \frac{Sc}{n}\right) e^{-m_2 z} + \frac{Sc^2}{n}\right] e^{-nt} \tag{35}$$

RESULTS AND DISCUSSIONS

The effects of Schmidt number (Sc), permeability parameter (k), Grashof number (Gr), modified Grashof number (Gm), rotational parameter (E), Magnetic parameter (M) and Prandtl number $(\Pr)$ on primary velocity u at $n = 0.4$, $t = 1.0$ and $\varepsilon = 0.02$ are numerically observed and are shown in Figs. (1) to (4).

Effects of modified Grashof number (Gm) and Magnetic parameter (M) on primary velocity are shown in Fig. (1). It is observed that the primary velocity decreases with an increase in the values of Magnetic parameter (M). Also it is noticed that the primary velocity increases near the plate for $0 < z < 1.0$, attains the maximum value near $z = 1.0$ and decreases rapidly for $z > 1$. For increasing values of modified Grashof number (Gm) the primary velocity increases. Here also we observed that the maximum velocity occurs near $z = 1.0$ and then decreases rapidly with increasing z ($z > 1$).

Fig.(2) represents the primary velocity profiles for various values of Schmidt number (Sc) and rotational parameter (E). It is observed that the primary velocity decreases with an increase in the values of Schmidt number (Sc) or rotational parameter (E). For each Schmidt number (Sc) or rotational parameter (E) the primary velocity increases near the plate as z increases, attains maximum velocity near the plate at $z = 0.5$. Then it decreases rapidly as the fluid moves for away from the plate at $z > 0.5$.

Fig.(3) display the effects of Grashof number (Gr) and Prandtl number $(\Pr)$. It is noticed that the primary velocity decreases with an increase in the values of Grashof number (Gr) and it increases with an increase in the values of Prandtl number $(\Pr)$. For each Grashof number (Gr) or Prandtl number $(\Pr)$ the primary velocity increases for $0 < z < 1.0$, attains the maximum velocity at $z = 1.0$ and then it decreases rapidly for $z > 1.0$.

The effects of permeability parameter (k) on primary velocity are shown in Fig.(4). It is observed that the primary velocity increases with an increase in values of permeability parameter (k). Also we observed for each permeability parameter (k), the primary velocity increases for $0 < z < 1.0$, attains the maximum velocity at $z = 1.0$ and then it decreases rapidly for $z > 1.0$.

Fig.(5) depicts the temperature profiles for different values of Prandtl number $(\Pr)$. It is observed that, owing to an increase in the values of the Prandtl number $(\Pr)$ the temperature decreases.

The effects of Schmidt number (Sc) on concentration profiles are shown in Fig.(6). It is observed that the concentration decreases with an increase in the values of Schmidt number (Sc).

In the absence of the Magnetic field, these results are in agreement with the results of N.P. Singh et al. [23].

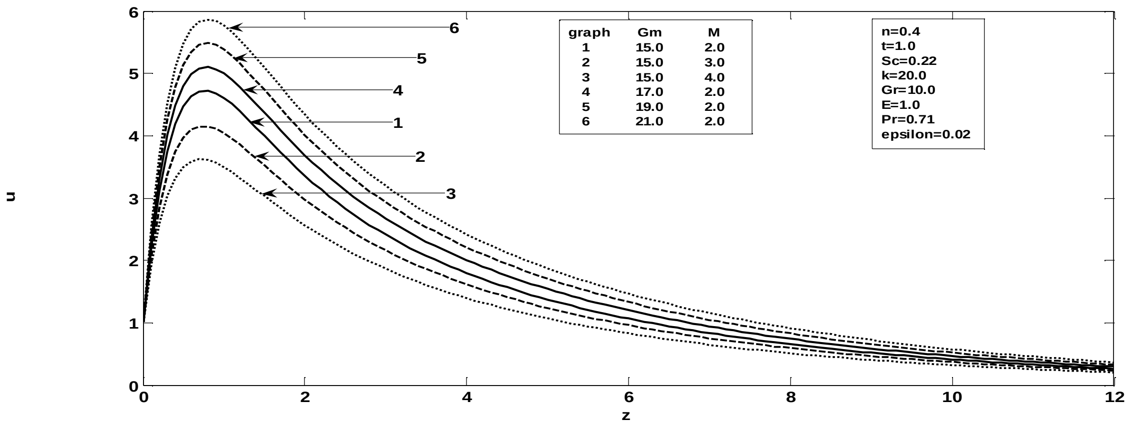

Fig.(1): The effects of Gm and M parameters on primary velocity

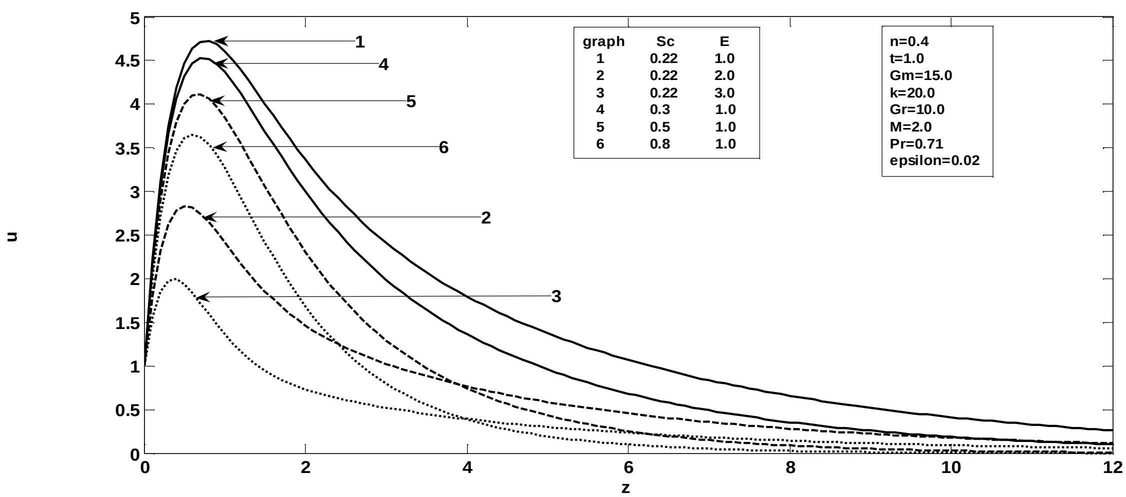

Fig.(2): The effects of Sc and E parameters on primary velocity

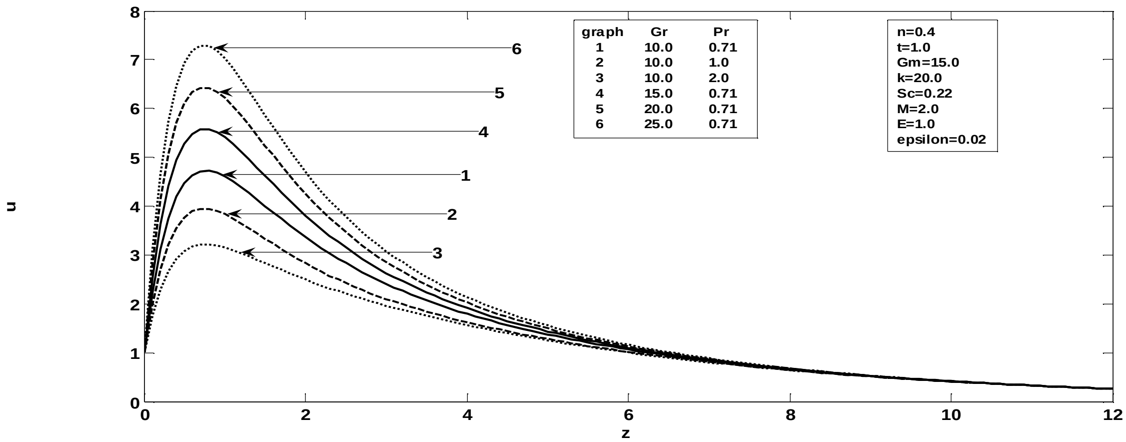

Fig.(3): The effects of Gr and $\Pr$ parameters on primary velocity

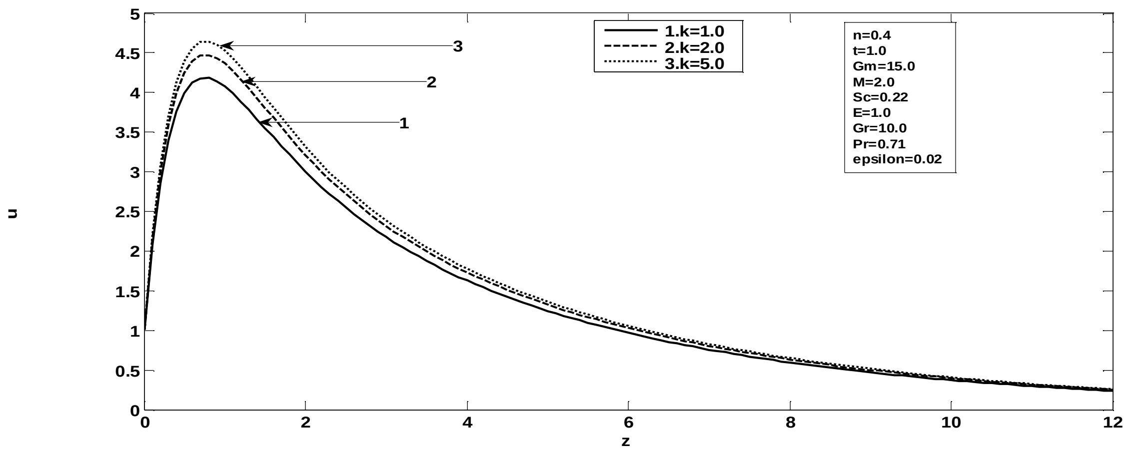

Fig.(4): The effects of permeability parameter k on primary velocity

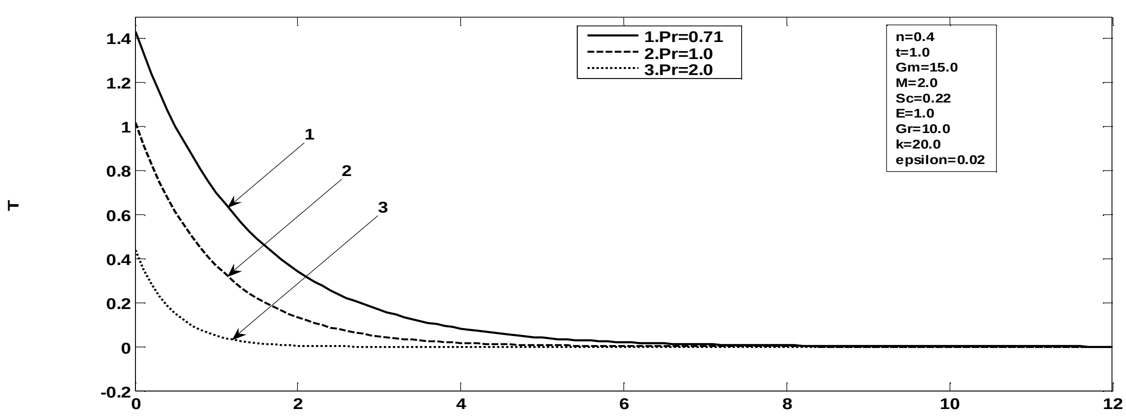

Fig.(5): The effects of Prandtl number $\Pr$ on Temperature

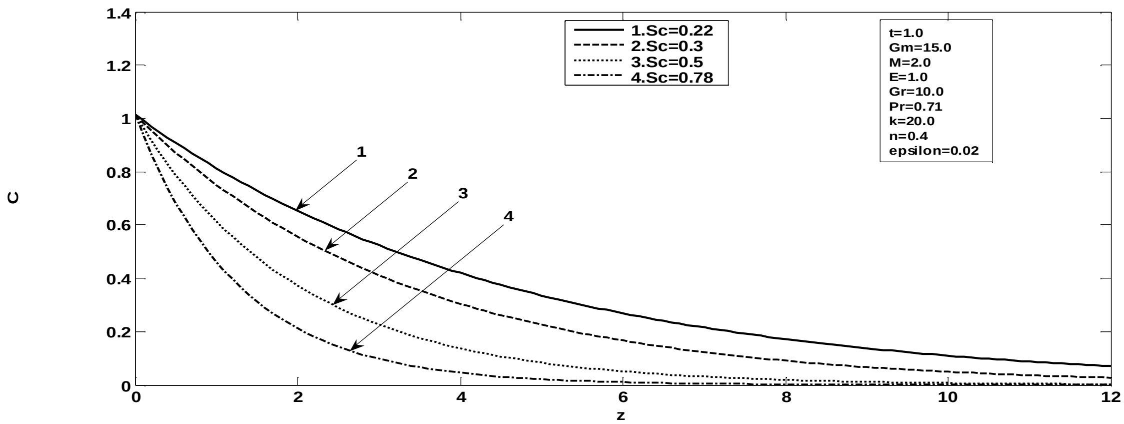

Fig.(6): The effects of Schmidt number Sc on Concentration

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APPENDIX

$$m_1 = \frac{1}{2}[\Pr + \sqrt{\Pr^2 - 4n\Pr}], \qquad m_2 = \frac{1}{2}[Sc + \sqrt{Sc^2 - 4nSc}]$$

$$A_1 = \frac{1}{2} + \frac{1}{2}\left[\frac{\sqrt{\left\{1+4\left(M+\frac{1}{k}\right)\right\}^2 + 64E^2} + \left\{1+4\left(M+\frac{1}{k}\right)\right\}}{2}\right]^{1/2}$$

$$B_1 = \frac{1}{2}\left[\frac{\sqrt{\left\{1+4\left(M+\frac{1}{k}\right)\right\}^2 + 64E^2} - \left\{1+4\left(M+\frac{1}{k}\right)\right\}}{2}\right]^{1/2}$$

$$A_2 = \frac{1}{2} + \frac{1}{2}\left[\frac{\sqrt{\left\{1+4\left(M+\frac{1}{k}-n\right)\right\}^2 + 64E^2} + \left\{1+4\left(M+\frac{1}{k}-n\right)\right\}}{2}\right]^{1/2}$$

$$B_2 = \frac{1}{2} + \frac{1}{2}\left[\frac{\sqrt{\left\{1+4\left(M+\frac{1}{k}-n\right)\right\}^2 + 64E^2} - \left\{1+4\left(M+\frac{1}{k}-n\right)\right\}}{2}\right]^{1/2}$$

$$A_5 = \Pr\left(\Pr^2 - \Pr - \frac{1}{k} - M\right), \qquad A_6 = \left(Sc^2 - Sc - \frac{1}{k} - M\right),$$

$$A_7 = \left(m_1^2 - m_1 - \frac{1}{k} - M + n\right), \qquad A_8 = \left(m_2^2 - m_2 - \frac{1}{k} - M + n\right)$$

$$A_9 = A_1^2 - B_1^2 - \frac{1}{k} - A_1 - M + n, \qquad A_{10} = 2A_1B_1 - B_1 - 2E,$$

$$A_{11} = \Pr^2 - \Pr - \frac{1}{k} - M + n, \qquad A_{12} = Sc^2 - Sc - \frac{1}{k} - M + n$$

$$P_1 = \frac{GrA_5}{A_5^2 + 4E^2\Pr^2}, \qquad P_2 = \frac{GmA_6}{A_6^2 + 4E^2}, \qquad Q_1 = \frac{2Gr\Pr E}{A_5^2 + 4E^2\Pr^2},$$

$$Q_2 = \frac{2GmE}{A_6^2 + 4E^2}, \qquad A_{13} = 1 + P_1 + P_2, \qquad A_{14} = Q_1 + Q_2,$$

$$P_3 = -A_1 A_{14} - B_1 A_{13}, \qquad P_4 = \frac{Gr}{n} + P_1 \Pr, \qquad P_5 = \frac{GmSc}{n} + P_2 Sc,$$

$$Q_3 = -A_1 A_{13} + B_1 A_{14}, \qquad Q_4 = Q_1 \Pr, \qquad Q_5 = Q_2 Sc, \qquad F_1 = \frac{Gr \Pr}{nm_1},$$

$$F_2 = Gm\left(1 - \frac{Sc}{n}\right), \qquad P_6 = \frac{F_1 A_7}{A_7^2 + 4E^2}, \qquad P_7 = \frac{F_2 A_8}{A_8^2 + 4E^2},$$

$$P_8 = \frac{P_3 A_9 + Q_3 A_{10}}{A_9^2 + A_{10}^2}, \qquad P_9 = \frac{P_4 A_{11} - 2EQ_4}{A_{11}^2 + 4E^2}, \qquad P_{10} = \frac{P_5 A_{12} - 2EQ_5}{A_{12}^2 + 4E^2},$$

$$P_{11} = 1 - P_6 + P_7 + P_8 + P_9 + P_{10},$$

$$Q_6 = \frac{2F_1 E}{A_7^2 + 4E^2}, \qquad Q_7 = \frac{2F_2 E}{A_8^2 + 4E^2}, \qquad Q_8 = \frac{Q_3 A_9 - P_3 A_{10}}{A_9^2 + A_{10}^2},$$

$$Q_9 = \frac{Q_4 A_{11} + 2EP_4}{A_{11}^2 + 4E^2}, \qquad Q_{10} = \frac{Q_5 A_{12} + 2EP_5}{A_{12}^2 + 4E^2}, \qquad Q_{11} = -Q_6 + Q_7 + Q_8 + Q_9 + Q_{10}.$$