International Journal of Advanced Computer and Mathematical Sciences ISSN 2230-9624.
Vol 2, Issue 1, 2011, pp 89-99 http://bipublication.com

ON THE HANKEL-CLIFFORD TYPE TRANSFORMATIONS OF GENERALIZED FUNCTIONS

B.B. Waphare, S. B. Gunjal
MAEER's MIT Arts, Commerce and Science College,
Alandi, Pune-412105.

ABSTRACT:

In this paper we have defined two new Hankel-Clifford type transformations. We have extended these transformations to certain spaces of generalized functions and have proved several results on inversion, uniqueness, boundedness and analyticity. Finally theory thus developed is used to solve Dirichlet problems.

Keywords: Hankel-Clifford type transformation, testing function space, distributional generalized transform.

2000 mathematics subject classifications: 46 F12.

1. Introduction: In recent past many authors have developed the theory of Hankel transformations of generalized functions which have applications in engineering and technology (see [2,3,4,9,12,13,15]).
The conventional Hankel transformation defined by

$$h_\mu\left\{f(x)\right\}(y)=\int_0^\infty \sqrt{xy}\,J_\mu(xy)\,f(x)\,dx\;\left(0<y<\infty\right) \tag{1.1}$$

Was extended by Zemanian [14] to certain generalized functions of slow growth through a generalization of Parseval's equation. Later on Koh and Zemanian [6] extended (1.1) to a class of generalized functions by the kernel method, which is a more general extension of (1.1). In Betancor [1] the Hankel-Clifford transformations of order $\mu\geq 0,$ defined by

$$\left(h_{\mu,1}\,f\right)(y)=y^\mu\int_0^\infty C_\mu(xy)\,f(x)\,dx \tag{1.2}$$

and

$$\left(h_{\mu,2}\,f\right)(y)=\int_0^\infty x^\mu\,C_\mu(xy)\,f(x)\,dx\,, \tag{1.3}$$

where C_μ is the Bessel-Clifford function of the first kind and order μ, has been extended to certain spaces of generalized functions and $C_\mu(z)=z^{-\mu/2}\,J_\mu\left(2\sqrt{z}\right)$.
Inspired by Betancor [1], we define following two Hankel-Clifford type transformations and extend it to certain spaces of generalized functions:

$$\begin{aligned}\left(h_{\alpha,\beta,1}\,f\right)(y)&=y^{\alpha-\beta}\int_0^\infty (xy)^{-(\alpha-\beta)/2}\,J_{\alpha-\beta}\left(2\sqrt{xy}\right)f(x)\,dx\\&=y^{(\alpha-\beta)/2}\int_0^\infty x^{-(\alpha-\beta)/2}\,J_{\alpha-\beta}\left(2\sqrt{xy}\right)f(x)\,dx\end{aligned} \tag{1.4}$$

and

$$\left(h_{\alpha,\beta,2} f\right)(y) = y^{-(\alpha-\beta)/2} \int_0^\infty x^{(\alpha-\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right) f(x)\,dx \qquad (1.5)$$

The theory of Hankel-Clifford transformations were developed by many authors (see [7,8,10]).

Now we construct the testing function space $H_{\alpha,\beta}$ of all complex valued smooth functions $\phi(x)$ defined on $I=(0,\infty)$, such that

$$\rho_{m,n}(\phi) = \sup_{x\in I} \left| x^m D^n \left(x^{-(\alpha-\beta)} \phi(x) \right) \right| < \infty$$

for every $m,n,\in \square$. $h_{\alpha,\beta,1}$ is an automorphism onto $H_{\alpha,\beta}$ for $(\alpha-\beta)\geq 0$. The distributional generalized transform $h'_{\alpha,\beta,1} f$ of f, as the adjoint of the classical transform.

$$\left\langle h'_{\alpha,\beta,1} f, \phi \right\rangle = \left\langle f, h_{\alpha,\beta,1}\phi \right\rangle, \text{ for every } \phi \in H_{\alpha,\beta} \qquad (1.6)$$

Following Socas [11], we can prove that $h_{\alpha,\beta,2}$ - transform is an automorphism onto $H_0 = S$. For every $f \in S'$, we define the distributional generalized transform $h'_{\alpha,\beta,2} f$ of f by

$$\left\langle h'_{\alpha,\beta,2} f, \phi \right\rangle = \left\langle f, h_{\alpha,\beta,2}\, \phi \right\rangle, \text{ for every } \phi \in S \qquad (1.7)$$

Following Chaudhary [2], we introduce $H_{a,\alpha,\beta}$ as the testing function space of all complex valued functions $\phi(x)$ defined on I such that

$$\sup_{x\in I} \left| e^{-ax} B^m_{\alpha,\beta}\, \phi(x) \right| < \infty, \text{ for every } m \in N,$$

where $B_{\alpha,\beta} = x^{-(\alpha-\beta)} D\, x^{3\alpha+\beta} D = x^{2(\alpha+\beta)-1}\left[xD^2 + (3\alpha+\beta) D \right]$.

The distributional generalized $h'_{\alpha,\beta,1}$- transform $h_{\alpha,\beta,1} f$ of f is defined as

$$\left(h'_{\alpha,\beta,1} f\right)(y) = \left\langle f(x), y^{\alpha-\beta} (xy)^{-(\alpha-\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right) \right\rangle, \text{ for every } y>0 \quad (1.8)$$

and for every $f \in H'_{a,\alpha,\beta}$.

In this paper we introduce two new spaces $L_{\alpha,\beta,a}$ and $Y_{\alpha,\beta,a}$ and we define the distributional generalized complex Hankel-Clifford type transformation on the dual space $L'_{\alpha,\beta,a}$ and $Y'_{\alpha,\beta,a}$ as

$$\left(h'_{\alpha,\beta,1} f\right)(y) = \left\langle f(x), x^{\alpha-\beta} (xy)^{-(\alpha-\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right) \right\rangle \text{ ,for every } f \in L'_{\alpha,\beta,a}, \quad (1.9)$$

and

$$\left(h'_{\alpha,\beta,2} f\right)(y) = \left\langle f(x), y^{\alpha-\beta} (xy)^{-(\alpha-\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right) \right\rangle, \text{ for every } f \in Y'_{\alpha,\beta,a} \qquad (1.10)$$

2. The spaces $L_{\alpha,\beta,a}$ and $L_{\alpha,\beta}(\sigma)$ and their duals:

Let $(a,\alpha-\beta)$ be a pair of positive real numbers. We define the space $L_{\alpha,\beta,a}$ of testing functions that consists of all smooth complex valued functions ϕ defined on $I=(0,\infty)$ such that

$$\eta^a_{\alpha,\beta,m}(\phi) = \sup_{x\in I} \left| e^{-a\sqrt{2x}}\, x^\beta\, B^{+m}_{\alpha,\beta}\, \phi(x) \right| < \infty, \text{ for } m \in \square,$$

where $B^+_{\alpha,\beta} = D\, x^{3\alpha+\beta} D\, x^{-(\alpha-\beta)} = x^{3\alpha+\beta-1}\left[(3\alpha+\beta) D + x D^2 \right] x^{-(\alpha-\beta)}$.

It can be easily shown that $\Gamma=\{\eta^{a}_{\alpha,\beta,m}\}_{m\in\mathbb{N}}$ is a separating collection of semi norms and $L_{\alpha,\beta,a}$ equipped with the topology generated by Γ is a countably multinormed space.
We shall require following properties of $L_{\alpha,\beta,a}$ in the sequel and we state them in the form of following lemmas :
Lemma 2.1: The space $L_{\alpha,\beta,a}$ is complete, therefore $L_{\alpha,\beta,a}$ is a Frechet space.
Lemma 2.2: The space $L_{\alpha,\beta,a}$ is a space of testing functions and its dual space, $L'_{\alpha,\beta,a}$ is a space of generalized functions.
Lemma 2.3: D(I) is contained in $L_{\alpha,\beta,a}$. The topology of D(I) is stronger than the topology induced in it by $L_{\alpha,\beta,a}$.
Lemma 2.4: $L_{\alpha,\beta,a}\subset E(I)$, the inclusion being continuous. Moreover, since D(I) is dense in E(I), $L_{\alpha,\beta,a}$ is dense in E(I). Hence we can consider the dual space of E(I), $E'(I)$, as a subspace of $L'_{\alpha,\beta,a}$.
Lemma 2.5: If $0<a_1<a_2$ then $L_{\alpha,\beta,a_1}\subset L_{\alpha,\beta,a_2}$ and the topology of $L_{\alpha,\beta,a}$ is stronger than the one induced in it by L_{α,β,a_2}.
Now we define countable union spaces.
Let $\{a_\nu\}_{\nu\in\mathbb{N}}$ be a monotonically increasing sequence of positive numbers tending to σ (possibly, $\sigma=\infty$). The space $L_{\alpha,\beta}(\sigma)=\bigcup_{\nu=1}^{\infty}L_{\alpha,\beta,a_\nu}$, denotes the countable space generated by the sequence of Frechet spaces $\{L_{\alpha,\beta,a_\nu}\}_{\nu=1}^{\infty}$. $L_{\alpha,\beta}(\sigma)$ is equipped with the usual topology.
We denote $L'_{\alpha,\beta}(\sigma)$ as dual of $L_{\alpha,\beta}(\sigma)$ with weak topology.
Lemma 2.6: $L_{\alpha,\beta}(\sigma)$ is a space of testing functions and $L'_{\alpha,\beta}(\sigma)$ is a space of generalized functions.
Lemma 2.7: The mapping $\phi\to B^{+}_{\alpha,\beta}\phi$ is linear and continuous of $L_{\alpha,\beta,a}$ into itself, and of $L_{\alpha,\beta}(\sigma)$ into itself.
We define the operator $B^{*}_{\alpha,\beta}$ on $L'_{\alpha,\beta,a}$ by

$$\langle B^{*}_{\alpha,\beta}\ f,\phi\rangle=\langle f,B_{\alpha,\beta}\,\phi\rangle\ ,\ \phi\in L_{\alpha,\beta,a}\ \text{and}\ f\in L'_{\alpha,\beta,a}$$

and in a similar way, $B^{*}_{\alpha,\beta}$ is defined on $L'_{\alpha,\beta}(\sigma)$.
Lemma 2.8: $B^{*}_{\alpha,\beta}$ is a continuous linear mapping of $L'_{\alpha,\beta,a}$ into itself and of $L'_{\alpha,\beta}(\sigma)$ into itself.
Following Zemanian [14], we can have
Lemma 2.9: For each $f\in L'_{\alpha,\beta,a}$ there exist a non-negative integer r and a positive constant C such that

$$|\langle f,\phi\rangle|\le C\max_{0\le m\le r}\tau^{a}_{\alpha,\beta,m}(\phi),\ \text{for all}\ \phi\in L_{\alpha,\beta,a}.$$

Lemma 2.10 : If f is a locally integrable function on I and if

$$\int_0^\infty \left| f(x) \right| e^{a\sqrt{2x}} x^{-\beta} dx < \infty,$$

then f generates a regular distribution in $L'_{\alpha,\beta,a}$ through the definition

$$\langle f, \phi \rangle = \int_0^\infty f(t)\phi(t)dt, \text{ for every } \phi \in L_{\alpha,\beta,a}.$$

Lemma 2.11: Let $x \in I = (0,\infty), a > 0$ and $(\alpha - \beta) \geq 0$. There exists a positive constant $A_{\alpha,\beta}$ such that

$$\left| e^{-a\sqrt{2x}} C_{\alpha,\beta}(xy) \right| \leq A_{\alpha,\beta},$$

for every y in the complete region

$$J_a = \left\{ y \in \mathbb{C} : \left| \operatorname{Im} \sqrt{2y} \right| < a, y \notin (-\infty, 0) \right\}.$$

By using Lemma 2.11, we immediately obtain the following

Lemma 2.12: Let $(\alpha - \beta) \geq 0$ and $\sigma > 0$. Then for $x \in (0,\infty)$, we have

$$\frac{\partial^m}{\partial y^m}\left[x^{\alpha-\beta} C_{\alpha,\beta}(xy) \right] \in L_{\alpha,\beta,a,} \text{ for each } m \in \mathbb{N} \text{ and } y \in J_a$$

Also

$$\frac{\partial^m}{\partial y^m}\left[y^m C_{\alpha,\beta}(xy) \right] \in L_{\alpha,\beta}(\sigma), \text{ for every } m \in N \text{ and } y \in J_\sigma.$$

Lemma 2.13: The space $H_{\alpha,\beta} \subset L_{\alpha,\beta,a}$ and the topology of $H_{\alpha,\beta}$ is stronger than the topology induced in it by $L_{\alpha,\beta,a}$. Also $H_{\alpha,\beta} \subset L_{\alpha,\beta}(\sigma)$ and the inclusion is continuous.

On the other hand, since the function $x^{\alpha-\beta} C_{\alpha,\beta}(xy)$ is not of rapid decent as $x \to \infty$, then $x^{\alpha-\beta} C_{\alpha,\beta}(xy)$ is not in $H_{\alpha,\beta}$. Hence $H_{\alpha,\beta}$ is strictly contained in $L_{\alpha,\beta,a}$.

Now we give a structure formula for the restriction of an element in $L'_{\alpha,\beta,a}$ to D(I).

Theorem 2.1: Let f be an arbitrary element of $L'_{\alpha,\beta,a}$. Then there exist bounded measurable functions $g_i(x)$ defined for $x > 0, i = 0,1,2...,r,$ where r is some non-negative integer depending on f, such that for every $\phi \in D(I)$,

$$\langle f, \phi \rangle = \left\langle \sum_{i=0}^{r} B^{*i}_{\alpha,\beta} \left\{ e^{-a\sqrt{2x}} x^\beta (-D) g_i(x) \right\}, \phi(x) \right\rangle.$$

Proof: Let $f \in L'_{\alpha,\beta,a}$. In view of Lemma 2.9, there exist a positive constant C and a non-negative integer r, such that

$$\left| \langle f, \phi \rangle \right| \leq C \max_{0 \leq i \leq r} \left\| D_x \left[e^{-a\sqrt{2x}} x^\beta B^{+i}_{\alpha,\beta} \phi(x) \right] \right\|_{L^1(0,\infty)},$$

for every $\phi \in D(I)$. Now by using Hahn-Banach Theorem and the Riesz representation theorem there exist bounded measurable functions g_i defined over $I = (0,\infty),\ i = 0,1,.....,r$ satisfying

$$\langle f,\phi\rangle=\left\langle \sum_{i=0}^{r} B^{*i}_{\alpha,\beta}\left[e^{-a\sqrt{2x}}\,x^{\beta}(-D)\,g_i(x)\right],\phi(x)\right\rangle .$$

for every $\phi\in D(I)$. Thus the proof is completed.

3. The generalized complex $h_{\alpha,\beta,1}$ - transformation:

Let $(\alpha-\beta)\geq 0$ and $a>0$. In view of Lemma 2.5, for every functional $f\in L'_{\alpha,\beta,a}$, there exists a unique real number of (possibly $\sigma_f=+\infty$) such that $f\in L'_{\alpha,\beta,b}$ if $b<\sigma_f$ and $f\notin L'_{\alpha,\beta,b}$ if $b>\sigma_f$. Thus $f\in L'_{\alpha,\beta}(\sigma)$. We define the generalized $h'_{\alpha,\beta,1}$-transformation $h'_{\alpha,\beta,1} f$ of f, by the relation.

$$F(y)=\left(h'_{\alpha,\beta,1} f\right)(y)=\left\langle f(x),x^{\alpha-\beta}\,C_{\alpha,\beta}(xy)\right\rangle ,\text{for every } y\in J_f, \qquad (3.1)$$

where

$$J_f=\left\{y\in\mathbb{C}:\left|\operatorname{Im}\sqrt{2}\,y\right|<\sigma_f,\ y\notin(-\infty,0]\right\}.$$

Note that as $x^{\alpha-\beta}\,C_{\alpha,\beta}(xy)\in L_{\alpha,\beta}(\sigma_f)$, for every $y\in J_f$, relation (3.1) is well defined.

Theorem 3.1: $F(y)$ as defined by (3.1) is an analytic function on y in the region of definition J_f and

$$D^n_y F(y)=\left\langle f(x),x^{\alpha-\beta}\,\frac{\partial^n}{\partial y^n}C_{\alpha-\beta}(xy)\right\rangle,\ \text{ for each } n\in\mathbb{N}.$$

Proof: Let $y\in J_f$. We can choose a real number a such that $y\in J_a\subset J_f$. Let C and C_1 denote two circles with respective radius r and r_1 such that $r<r_1$. Moreover these circles lie completely within σ_f. Let Δ_y be a non-zero complex increment such $|\Delta y|<r$.

Now consider

$$\frac{F(y+\Delta y)-F(y)}{\Delta y}-\left\langle f(x),\ x^{\alpha-\beta}\,\frac{\partial}{\partial y}C_{\alpha,\beta}(xy)\right\rangle=\left\langle f(x),\Psi_{\Delta y}(x,y)\right\rangle,$$

where,

$$\Psi_{\Delta y}(x,y)=x^{\alpha-\beta}\left\{\frac{C_{\alpha,\beta}\left(x(y+\Delta y)\right)-C_{\alpha,\beta}(xy)}{\Delta y}-\frac{\partial}{\partial y}C_{\alpha,\beta}(xy)\right\}$$

Our aim is to show that $\Psi_{\Delta y}(x,y)$ converges to zero as $\Delta y\to 0$ in $L_{\alpha,\beta,a}$. By using the Cauchy integral formula and by interchanging $\frac{\partial}{\partial y}$ with $B^{+m}_{\alpha,\beta}$, we can write

$$B^{+m}_{\alpha,\beta,x}\left\{\Psi_{\Delta y}(x,y)\right\}=\frac{1}{2\pi i}\int_{C_1}\frac{\Delta y}{(\eta-y)^2(\eta-y-\Delta y)}(-1)^m\,\eta^{m+\alpha-\beta}\,x^{\alpha-\beta}\,C_{\alpha,\beta}(x\eta)\,d\eta,$$

for every $m\in\mathbb{N}$.

By Lemma 2.11, we have:

$$\sup_{x\in I}\left|e^{-a\sqrt{2x}}\,\frac{x^{\beta}}{2\pi i}\int_{C_1}\frac{\Delta y}{(\eta-y)^2(\eta-y-\Delta y)}(-1)^m\,\eta^{m+\alpha-\beta}\,x^{\alpha-\beta}\,C_{\alpha,\beta}(xy)\,d\eta\right|\leq K_{\alpha,\beta}|\Delta y|\to 0, as\,\Delta y\to 0,$$

where $K_{\alpha,\beta}$ is a suitable positive constant.

Thus, $\eta^{a}_{\alpha,\beta,m}\left(\Psi_{\Delta y}\left(x,y\right)\right)\to 0,\ as\,\Delta y\to 0,$ for every $m\in\mathbb{N}$. Thus

$$\lim_{\Delta y\to 0}\frac{F\left(y+\Delta y\right)-F\left(y\right)}{\Delta y}=\left\langle f\left(x\right),x^{\alpha-\beta}\frac{\partial}{\partial y}C_{\alpha,\beta}\left(xy\right)\right\rangle .$$

Thus the proof is completed by using induction on n

Theorem 3.2: Let $F\left(y\right)$ be the generalized $h'_{\alpha,\beta,1}$- transform of $f\in L'_{\alpha,\beta,a}$. $F\left(y\right)$ is bounded according to

$$\left|F\left(y\right)\right|\le P_a\left(\left|y\right|\right),$$

for every

$$y\in J_a=\left\{y\in\mathbb{C}:\left|\operatorname{Im}\sqrt{2y}\right|<a<\sigma_f,\ y\notin\left(-\infty,0\right]\right\},$$

where P_a is a polynomial depending on a.

Moreover, $F\left(y\right)$ satisfies the inequality :

$$\left|F\left(y\right)\right|\le\begin{cases}C & ,\ for\ \ 0<\left|y\right|<1\\ C\left|y\right|^{r} & ,\ for\ \ \left|y\right|>1,\end{cases}$$

where C is a positive constant and r is a non-negative integer.

Now we show that the generalized $h'_{\alpha,\beta,1}$- transformation of f in $L'_{\alpha,\beta}\left(\sigma_f\right)$ given by (3.1) coincides (in the sense of equality in $H'_{\alpha,\beta}$) with the generalized $h'_{\alpha,\beta,1}$- transformation of f as given by (1.6). This statement cannot be proved if definition (1.8) is adopted.

Theorem 3.3: Let $f\in L'_{\alpha,\beta}\left(\sigma_f\right),\psi\in H_{\alpha,\beta}$ and $\left(\alpha-\beta\right)\ge 0.$ Then

$$\left\langle\left\langle f\left(x\right),x^{\alpha-\beta}\,C_{\alpha,\beta}\left(xy\right)\right\rangle,\psi\left(y\right)\right\rangle=\left\langle f\left(x\right),x^{\alpha-\beta}\int_0^\infty C_{\alpha-\beta}\left(xy\right)\psi\left(y\right)dy\right\rangle.$$

Proof: We now restrict y to the positive real line. In view of Theorem 3.2, the function $F\left(y\right)=\left\langle f\left(x\right),x^{\alpha-\beta}\,C_{\alpha,\beta}\left(xy\right)\right\rangle$ generates a regular element in $H'_{\alpha,\beta}$ by the relation

$$\left\langle F\left(y\right),\psi\left(y\right)\right\rangle=\int_0^\infty F\left(y\right)\psi\left(y\right)dy,\ \text{for every}\ \psi\in H_{\alpha,\beta}.$$

Hence

$$\left\langle\left\langle f\left(x\right),x^{\alpha-\beta}\,C_{\alpha,\beta}\left(xy\right)\right\rangle,\psi\left(y\right)\right\rangle=\int_0^\infty\left\langle f\left(x\right),x^{\alpha-\beta}\,C_{\alpha,\beta}\left(xy\right)\psi\left(y\right)\right\rangle dy,$$

$\psi\in H_{\alpha,\beta}$. To conclude the proof, we need to establish the following:

$$\int_0^\infty\left\langle f\left(x\right),x^{\alpha-\beta}\,C_{\alpha,\beta}\left(xy\right)\psi\left(y\right)\right\rangle dy=\left\langle f\left(x\right),x^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta}\left(xy\right)\psi\left(y\right)dy\right\rangle.$$

This we can prove by using the technique of Riemann Sums.

Let $0<a<b<\infty$ and let $\left\{y_\nu^{(n)}\right\}_{\nu=1}^{n}$ be a partition of the interval $\left(a,b\right)$ such that if we denote $d_n=y_\nu^{(n)}-y_{\nu-1}^{(n)},\ \ \nu=2,3,....,n,$ then $d_n\to 0$ as $n\to\infty.$ We can obtain that

$$\sup_{x\in I}\left| e^{-a\sqrt{2x}}\, x^{\beta}\, B^{+m}_{\alpha,\beta,x}\, x^{\alpha-\beta}\left\{\int_a^b C_{\alpha,\beta}(xy)\psi(y)\,dy - dn\sum_{\nu=1}^{n} C_{\alpha,\beta}\left(x, y_\nu^{(n)}\right)\psi\left(y_\nu^{(n)}\right)\right\}\right| \to 0,$$

as $n\to\infty$.

Thus we have

$$\int_a^b \left\langle f(x), x^{\alpha-\beta} C_{\alpha,\beta}(xy)\psi(y)\right\rangle dy = \left\langle f(x), x^{\alpha-\beta}\int_a^b C_{\alpha,\beta}(xy)\psi(y)\,dy\right\rangle.$$

Now, by letting $b\to\infty$ and $a\to 0^+$, the proof of the theorem is completed.

Theorem 3.4 (Inversion): Let $f\in L'_{\alpha,\beta}(\sigma_f)$ and $F(y)$ be the $h'_{\alpha,\beta,1}$- transform of f. Then

$$\lim_{N\to\infty}\left\langle \int_0^N F(y)\, y^{\alpha-\beta}\, C_{\alpha,\beta}(xy)\,dy, \psi(x)\right\rangle = \left\langle f(x), \psi(x)\right\rangle, \quad \text{for every } \psi\in D(I).$$

Proof: Let $\psi\in D(I)$. According to Theorem 3.2 and the fact that the support of Ψ is a compact subset of I we may write by using Fubini's Theorem:

$$\left\langle \int_0^N F(y)\, y^{\alpha-\beta}\, C_{\alpha,\beta}(xy)\,dy, \psi(x)\right\rangle = \int_0^\infty \psi \int_0^N F(y)\, y^{\alpha-\beta}\, C_{\alpha,\beta}(xy)\,dy\,dx$$

$$= \int_0^N \left\langle f(x), x^{\alpha-\beta} C_{\alpha,\beta}(xy)\right\rangle y^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta}(xy)\psi(x)\,dxdy,$$

for every $N>0$.

Again by using the technique of Riemann sums, we get

$$\left\langle \int_0^N F(y)\, y^{\alpha-\beta}\, C_{\alpha,\beta}(xy)\,dy, \psi(x)\right\rangle = \left\langle f(x), x^{\alpha-\beta}\int_0^N C_{\alpha,\beta}(xy)\phi(y)\,dy\right\rangle,$$

where $\phi(y) = h_{\alpha,\beta,1}\{\psi(x)\}(y)$.

Now set

$$G_N(t,x) = \int_0^N C_{\alpha,\beta}(ty)\, C_{\alpha,\beta}(xy)\, y^{\alpha-\beta}\,dy.$$

Note that for positive numbers a and b, we have

$$\lim_{N\to\infty}\int_a^b G_N(t,x)\, x^{\alpha-\beta}\,dx = \begin{cases} 1, & for\ t\in(a,b) \\ \frac{1}{2}, & for\ t=a\ or\ t=b \\ 0, & for\ t\notin[a,b], \end{cases}$$

by virtue of the inversion formula for the classical $h_{\alpha,\beta,1}$-transform (see Mendez [7]) since

$$B^{+m}_{\alpha,\beta,t}\left\{ t^{\alpha-\beta}\int_a^b G_N(t,x)\psi(x)\,dx - \psi(t)\right\}$$

$$= t^{\alpha-\beta}\int_a^b x^{\alpha-\beta}\, G_N(t,x)\left\{\psi_m(x) - \psi_m(t)\right\}dx,$$

where $\psi_m(x) = B^{+m}_{\alpha,\beta,x}\left\{x^{-(\alpha-\beta)}\psi(x)\right\}$, fro every $m\in\mathbb{N}$, the asymptotic behaviors of the Bessel-Clifford function $C_{\alpha,\beta}$ (see Hayek [5]) enable us to show that for any $c>0$

$$e^{-c\sqrt{2t}}\, t^{\alpha} \int_a^b x^{\alpha-\beta}\, G_N(t,x)\{\psi_m(x)-\psi_m(t)\}\, dx \to 0,$$

as $N\to\infty$ Uniformly on $t\in(0,\infty)$, for every $\Psi\in D(I)$ whose support is contained in [a,b].

Hence $x^{\alpha-\beta}\int_0^N C_{\alpha,\beta}(xy)\phi(y)\,dy \to \psi(x)$, as $N\to\infty$, in the sense of convergence in $L_{\alpha,\beta}(\sigma_f)$. This completes the proof.

Corollary 3.5 (Week version of uniqueness Theorem):

Let $F(y)=(h'_{\alpha,\beta,1} f)(y)$, for $y\in J_f$ and let $G(y)=(h'_{\alpha,\beta,1}g)(y)$ for $y\in J_g$, where J_f and J_g are the region of definitions of F and G respectively and f and g are in $L_{\alpha,\beta}(\sigma)$. If $F(y)=G(y)$ on $J_f\cap J_g$, then $f=g$ in the sense of equality in $D'(I)$.

Proof: Proof follows from the inversion Theorem 3.4 above.

4. The generalized transform $h'_{\alpha,\beta,2}$:

Definition 4.1: A function $\psi(x)\in Y_{\alpha,\beta,a}$ if and only if

(i) It is defined on $I=(0,\infty)$

(ii) it is complex valued and smooth and for each non-negative integer m

$$\gamma^m_{\alpha,\beta,a}(\psi)=\sup_{x\in I}\left|e^{-a\sqrt{2x}}\, x^{\alpha}\, B^{+m}_{\alpha,\beta}\psi(x)\right|<\infty.$$

Note that $Y_{\alpha,\beta,a}$ endowed with the topology generated by the multinorm $\Gamma=\{\gamma^m_{\alpha,\beta,a}\}_{m\in\mathbb{N}}$ is a space of testing functions. $Y'_{\alpha,\beta,a}$ denotes the dual of $Y_{\alpha,\beta,a}$ which is a space of generalized functions.

Note that $D(I)\subseteq S\subset Y_{\alpha,\beta,a}\subset E(I)$.

For every $y\in J_a=\{y\in\not\subset:|I_m\sqrt{2y}|<a,\ y\neq 0$ y non-negative number, the function

$$\frac{\partial^m}{\partial y^m}\{y^{\alpha-\beta}\, C_{\alpha,\beta}(xy)\}\in Y_{\alpha,\beta,a}\,.$$

Also if $\{a_\nu\}_{\nu\in N}$ is a monotonically increasing sequence of positive numbers tending to σ (possibly $\sigma=\pm\infty$), the countable union space.

$$Y_{\alpha,\beta}(\sigma)=\bigcup_{\nu=1}^{\infty} Y_{\alpha,\beta,a_\nu}$$

can be defined.

It can be seen that if $f\in Y'_{\alpha,\beta,a}$ for some $a>0$, then there exists a positive real number σ_f such that $f\notin Y'_{\alpha,\beta,b}$ for $b>\sigma_f$ and $f\in Y'_{\alpha,\beta,b}$ for every $b<\sigma_f$; hence $f\in Y'_{\alpha,\beta}(\sigma_f)$. Now we define the generalized $h'_{\alpha,\beta,2}$- transform $h'_{\alpha,\beta,2}\, f$ of f as

$$F(y)=(h'_{\alpha,\beta,2}\, f)(y)=\langle f(x), y^{\alpha-\beta}\, C_{\alpha,\beta}(xy)\rangle,$$

if $(\alpha-\beta)\geq 0$ and $y\in J_f=\{y\in\mathbb{C}:|\mathrm{Im}\sqrt{2y}|<\sigma_f, y\notin(-\infty,0\,]\}$.

Theorem 4.2: Let $f \in Y'_{\alpha,\beta}(\sigma_f)$ and $F(y) = (h'_{\alpha,\beta,2} f)(y)$ for $y \in J_f$. Then $F(y)$ is an analytic function on J_f, and $D_y^n F(y) = \left\langle f(x), \frac{\partial^n}{\partial y^n}\{y^{\alpha-\beta} C_{\alpha,\beta}(xy)\}\right\rangle, n \in \mathbb{N}$, $y \in J_f$

Proof: Proof is simple and hence omitted.

Theorem 4.3: $F(y)$ is bounded in any region J_a according to

$$|F(y)| \le |y|^{\alpha-\beta} P_a(|y|),$$

where P_a is a polynomial depending on a and J_a denotes the same region as that in former sections.

Theorem 4.4: Let f be an element of $Y'_{\alpha,\beta}(\sigma)$ and $(\alpha-\beta) \ge 0$. If ψ is in S, then

$$\langle F(y), \psi(y)\rangle = \langle f(x), h_{\alpha,\beta,2}\{\psi(y)\}(x)\rangle,$$

where $F(y) = \langle f(x), y^{\alpha-\beta} C_{\alpha,\beta}(xy)\rangle$.

Theorem 4.5: Let $f \in Y'_{\alpha,\beta}(\sigma)$ and let $F(y)$ be the $h'_{\alpha,\beta,2}$- transformation of f.

Then

$$\lim_{N\to\infty}\left\langle \int_0^N F(y) x^{\alpha-\beta} C_{\alpha,\beta}(xy)\, dy, \psi(x)\right\rangle = \langle f, \psi\rangle,$$

for every $\psi \in D(I), (\alpha-\beta) \ge 0$.

Theorem 4.6: Let $F(y) = (h'_{\alpha,\beta,2} f)(y)$ for $y \in J_f$, let $G(y) = (h'_{\alpha,\beta,2} g)(y)$ for $y \in J_g$, and assume that $F(y) = G(y)$ for $y \in J_f \cap J_g$. Then, $f = g$, in the sense of equality in $D'(I)$.

5. Applications:

Consider the distributional differential equations of the type

$$P(B)u = g$$

where $B = B^+_{\alpha,\beta}$ or $B = B^+_{\alpha,\beta}$, P is a polynomial, u and g are transformable functionals.The following theorem is of interest.

Theorem 5.1: Let P be a polynomials, then

$$\left(h'_{\alpha,\beta,1} P\left(B^*_{\alpha,\beta}\right)u\right)(y) = P(-y)\left(h'_{\alpha,\beta,1} u\right)(y), \text{ for each } u \in L'_{\alpha,\beta}(\sigma)$$

$$\left(h'_{\alpha,\beta,2} P\left(B^+_{\alpha,\beta}\right)v\right)(y) = P(-y)\left(h'_{\alpha,\beta,2} v\right)(y), \text{ for each } v \in Y'_{\alpha,\beta}(\sigma).$$

The above results can be deduced from the equality

$$B^+_{\alpha,\beta,x}\left[x^{\alpha-\beta} C_{\alpha,\beta}(xy)\right] = -y\, x^{\alpha-\beta} C_{\alpha,\beta}(xy).$$

Now we will discuss the integral transformations so far we have defined in this paper, which are useful in the solution of Dirichlet problems.

We state the first problem as follows:

To find a conventional function $v(r,z)$ on the domain $\{(r,z) : 0 < r < \infty, 0 < z < \infty\}$, that satisfies the equation

$$\frac{\partial^2 v(r,z)}{\partial z^2} + B^+_{\alpha,\beta,r} v(r,z) = 0, \quad \text{for } (\alpha-\beta) \ge 0 \tag{5.1}$$

with the following boundary conditions:

(i) $v(r,z)$ converges to $f \in Y'_{\alpha,\beta}(\sigma)$, as $z \to 0^+$, in the sense of convergence in $D'(I)$

(ii) $r^{-(\alpha-\beta)/2} v(r,z) = 0\left(r^{-\frac{1}{4}}\right), as\ r \to \infty$

(iii) $v(r,z) \to 0$, as $z \to \infty$ uniformly on $r \in (0,\varepsilon)$ for every $\varepsilon > 0$.

(iv) If $(\alpha-\beta) > 0$, $v(r,z) \to 0$ as $r \to 0^+$, for every $z > 0$, and if $(\alpha-\beta) = 0$, $v(r,z)$ is bounded on $0 < r < \infty$, for every $z > 0$.

We adhere to customary technique of first deriving the solution formally and leaving as a subsequent step the proof that the result we have obtained truly satisfies the differential equation (5.1) and the boundary conditions (i) – (iv).

A formal application of the $h_{\alpha,\beta,1}$ - transformation and the boundary conditions leads to

$$v(r,z) = r^{\alpha-\beta} \int_0^\infty C_{\alpha,\beta}(rt)\, e^{-\sqrt{t}\,z} \left\langle f(x), t^{\alpha-\beta} C_{\alpha,\beta}(tx) \right\rangle dt. \tag{5.2}$$

The relation (5.2) is truly the solution we seek can be shown by using Lebesgue's dominated convergence and Riemann-Lebsgue Lemma. In a similar way by applying the $h_{\alpha,\beta,2}$ - transformation, we can tackle the following Dirichlet problem. Now consider the new differential equation

$$\frac{\partial^2 v(r,z)}{\partial z^2} + B^+_{\alpha,\beta,\gamma}\, v(r,z) = 0, \text{ for } (\alpha-\beta) \geq 0$$

with the boundary conditions:

(i) $v(r,z)$ converges to the generalized function $f \in L'_{\alpha,\beta}(\sigma)$ in the sense of convergence in $D'(I)$;

(ii) $v(r,z) \to 0$ as $z \to \infty$, uniformly on $r \in (0,\infty)$,

(iii) $r^{(\alpha-\beta)/2} v(r,z) = 0\left(r^{-\frac{1}{4}}\right)$, as $\gamma \to \infty$,

(iv) $v(r,z)$ is bounded on $0 < r < \infty$, for every $z > 0$.

REFERENCES:

1. Betancor J.J., Two complex variants of a Hankel type transformation of generealized functions, Portugaliac Mathematica Vol. 46, Fasc 3, (1989),229-243.
2. Chaudhary M.S., Hankel type transform of distributions, Ranchi Univ. Math. Jour. V. 12 (1981), 9-16.
3. Gelfand I.M. and Shilov G.E., Les Distributions, Tome 2, Dunod , Paris (1964).
4. Gray A., Mathews, G.B. and McRobert, T.M. – A treatise on Bessel functions and their applications to physics, McMillan and Co. Ltd. London (1952).
5. Hayek N., Estudio de la ecuacion differential $xy'' + (v+1)y' + y = 0$ y des sus aplicaciones, Colle Ctanea Mathematica 18 (1966-67), 55-174.

6. Koh E.L. and Zemanian A.H., The complex Hankel and I-transformations of generalized functions, SIAM J. Appl.Math. Vol. 16, No.5 (1968), 945-957.
7. Mendez J.M., La transformacion integral de Hankel- Clifford, La Laguna: Seoretariado de publications de La Universidad de La Laguna (1979).
8. Mendez J.M. and Socas M.M. – A generalized Hankel Clifford transformation (to appear).
9. Schwartz, L. – Theorie des distributions, Ed. Hermann, Paris (1959).
10. Socas M.M. and Mandez J.M., A Pair of generalized Hankel-Clifford Transformations and their applications (to appear)
11. Socas M.M. – La transformacion de Hankel-Clifford en ciertos espacios de funciones generalizadas, Doctoral Thesis, Dept. de Analisis Matematico de la Universidad de La Laguna (1986).
12. Watson G.N., A treatise on the theory of Bessel functions, Cambridge University Press (1958).
13. Zemanian A.H., A distributional K-transformations, J. SIAM Appl.Math. 14, No.6 (1966), 1350-1465.
14. Zemanian A.H.- Generalized Integral Transformation, Inter Science Publishers, New York (1968) ; republished by Dover Publications, New York (1987).
15. Zitomirski J.A., On differential operators of infinite order in spaces of type S, Math. USSR Sbornik, 9, No.3. (1969), 1379-1388.