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A NEW EXTENSION OF THE FINITE HANKEL TYPE TRANSFORM

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ABSTRACT: In this paper we have introduced a generalized finite Hankel type transform involving Bessel type functions as kernel. Inversion formula is established and some properties are given. These transform can be used to solve certain class of mixed boundary value problems. Unsteady flow through a concentric annulus has been considered as an application of this generalized finite Hankel type transform.

Keywords: Hankel type transform, generalized finite Hankel type transform, finite integrals.

1. Introduction: In recent past, transforms like Laplace, Hankel, Fourier, and Mellin have been applied to the solution of boundary value problems in physics, mathematics, engineering and technology. The applications of such type of transforms often reduces to a partial differential equation in n independent variables to one $n-1$ variables and it is often possible ,by successive operations of this type to reduce the problem to the solution of an ordinary differential equation [10].

In most of the problems in which the range of a variable is considered as $(0,\infty)$ .In applying the method of integral transforms to problems formulated on finite domains it is necessary to introduce finite integrals on the transform integral. Transforms of this nature are called finite transforms afford a more convenient method of solution than the methods which often require much ingenuity in assuming at the outset a correct solution form [3].A Bessel function as a kernel of a finite integral which Sneddon [8] defined as a Hankel transform and preceded to show its usefulness in revolving certain boundary value problems. In Cinelli[4],he has extended the finite Hankel transform method to include all symmetric and asymmetric cases for two surfaces using Bessel kernels of arbitrary order.Ali and Kalla [1] introduced a generalized form of the infinite Hankel transform and applied it to the problem of a heavy pollutant from a ground level aerial source within the framework of diffusion theory.

The purpose of the present paper is to introduce a generalized finite Hankel type transform of which the finite Hankel type transform of Sneddon[9] will come out as particular cases, and thus to extend its utility to a wider class of partial differential equations.

2. Establishment of transforms: A fairly number of differential equations occurring in physics and engineering problems is specialization of the form [2]

$$D_x^2 y+\frac{(1-2r)}{x}D_x y+\left[t^2+\frac{(r^2-\alpha^2-\beta^2+2\alpha\beta)}{x^2}\right]y(x)=0,(\alpha-\beta)\geq 0,t>0, \quad 2.1)$$

where $D_x=\frac{d}{dx}$.

The general solution of (2.1) is

$$y(x)=x^r\left[C_1 J_{\alpha-\beta}(tx)+C_2 Y_{\alpha-\beta}(tx)\right], \quad (2.2)$$

where C_1 and C_2 are arbitrary constants and $J_{\alpha-\beta}(tx)$ and $Y_{\alpha-\beta}(tx)$ are Bessel type functions of first and second kind respectively.
The main purpose of this paper is to generalize the finite Hankel transform by using (2.1) in which Bessel's equation is a special case of it when $\alpha = 1/4 + \mu/2, \beta = 1/4 - \mu/2$ and $r = 0$ .This work enables the technique to be applied to a wider class of problems.

3. Generalized finite Hankel type transform: The generalized finite Hankel type transform defined by the linear operator

$$H_{\alpha,\beta}[f(x)] = \int_0^a x^{1-r} f(x) J_{\alpha-\beta}(t_n x) dx = \bar{f}(t_n) \qquad (3.1)$$

where, $f(x)$ belongs to a certain class of functions for which the integral exists and $0 \le x \le a$ .If $t_n (n = 1, 2, 3....)$ are positive roots of the transcendental equation

$$J_{\alpha-\beta}(t_n a) = 0, \qquad (3.2)$$

the corresponding inversion formula is

$$f(x) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{\bar{f}(n)}{\left[J'_{\alpha-\beta}(t_n a) \right]^2} \left[x^r J_{\alpha-\beta}(t_n x) \right] . \qquad (3.3)$$

When $\alpha = 1/4 + \mu/2, \beta = 1/4 - \mu/2$ and $r = 0$,the above relations reduce to the case studied by [8].

If the field of variation of the variable x is an interval $a \le x \le b$ which doesn't include the origin, then we use the second type transform defined by

$$H_{\alpha,\beta}[f(x)] = \int_a^b x^{1-r} f(x) \left[J_{\alpha-\beta}(t_n x) Y_{\alpha-\beta}(t_n b) - J_{\alpha-\beta}(t_n b) Y_{\alpha-\beta}(t_n x) \right] dx = \bar{f}(t_n), \qquad (3.4)$$

where t_n are the positive roots of the transcendental equation

$$J_{\alpha-\beta}(t_n b) Y_{\alpha-\beta}(t_n a) - J_{\alpha-\beta}(t_n a) Y_{\alpha-\beta}(t_n b) = 0, \qquad (3.5)$$

then the corresponding inversion formula is

$$f(x) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{t_n^2 J_{\alpha-\beta}^2(t_n b) \bar{f}(n)}{J_{\alpha-\beta}^2(t_n b) - J_{\alpha-\beta}^2(t_n b)} \left[x^r \left\{ J_{\alpha-\beta}(t_n x) Y_{\alpha-\beta}(t_n b) - J_{\alpha-\beta}(t_n b) Y_{\alpha-\beta}(t_n x) \right\} \right] \qquad (3.6)$$

In the application of the finite Hankel type transforms to physical problems,it is useful to have available some formulae connecting the finite Hankel type transforms of derivatives of functions. For example,

$$\begin{aligned} & H_{\alpha,\beta}\left[D_x^2 + \frac{(1-2r)}{x} D_x + \frac{(r^2 - \alpha^2 - \beta^2 + 2\alpha\beta)}{x^2} \right] \\ & = \frac{2}{\pi^2 b^r} f(b) - \frac{2 J_{\alpha-\beta}(t_n b)}{\pi a^r J_{\alpha-\beta}(t_n a)} f(a) - t_n^2 \bar{f}(n) \end{aligned} \qquad (3.7)$$

If t_n are the roots of transcendental equation

$$J_{\alpha-\beta}(t_n b)\left[t_n Y'_{\alpha-\beta}(t_n a) + \left((r/a) + h \right) Y_{\alpha-\beta}(t_n a) \right] - \left[t_n J'_{\alpha-\beta}(t_n a) + \left((r/a) + h \right) J_{\alpha-\beta}(t_n a) \right] Y_{\alpha-\beta}(t_n b) = 0, \qquad (3.8)$$

then the corresponding inversion formula is

$$f(x)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{t_n^2\left[t_nJ'_{\alpha-\beta}(t_na)+((r/a)+h)J_{\alpha-\beta}(t_na)\right]^2\bar{f}(n)}{\left\{\left[t_nJ'_{\alpha-\beta}(t_na)+((r/a)+h)J_{\alpha-\beta}(t_na)\right]^2-\left(((r/a)+h)^2+\left[t_n^2-\frac{\alpha^2+\beta^2-2\alpha\beta}{a^2}\right]\right)J^2_{\alpha-\beta}(t_nb)\right\}} \quad (3.9)$$

$$\times\left\{x^r\left[J_{\alpha-\beta}(t_nx)Y_{\alpha-\beta}(t_nb)-J_{\alpha-\beta}(t_nb)Y_{\alpha-\beta}(t_nx)\right]\right\},$$

If we take $\alpha=1/4+\mu/2, \beta=1/4-\mu/2$ and $r=0$ in (3.9), then the case studied by Kalla [6] will come out as particular case of (3.9).

4. Application to hydrodynamics: Here we consider the flow through a concentric annulus when one of the cylinders starts rotating impulsively with a uniform angular velocity, while the other is kept fixed.It is assumed that the rate suction at one wall is equal to the rate of injection at the other.The Navier-Stokes equations and the equation of continuity reduce to Nada[7]

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial l^2}+\frac{1+R}{l}\frac{\partial v}{\partial l}-\frac{1-R}{l^2}v, \quad (4.1)$$

the boundary condition to be satisfied are

$$v(r,t)=1, l=1, \quad (4.2a)$$

$$v(l,t)=0, l=\sigma, \quad (4.2b)$$

$$v(l,t)=0, t=0. \quad (4.2c)$$

Now applying Laplace transform to (4.1) we get

$$\frac{d^2\bar{v}}{dl^2}+\frac{1+R}{l}\frac{d\bar{v}}{dl}-\frac{1-R}{l^2}\bar{v}=s\bar{v}, \quad (4.3)$$

we make use of the generalized finite Hankel type transform with $r=-(R/2)$ and $\alpha-\beta=(R/2)-1$.

Let

$$H_{\alpha,\beta}[v]=\bar{v}^*=\int_1^{\sigma} l^{1+R/2}\bar{v}[J_{\alpha-\beta}(la_k)Y_{\alpha-\beta}(\sigma a_k)-J_{\alpha-\beta}(\sigma a_k)Y_{\alpha-\beta}(la_k)]dl, \quad (4.4)$$

where a_k are chosen to be the positive roots of

$$J_{\alpha-\beta}(\sigma a_k)Y_{\alpha-\beta}(a_k)-J_{\alpha-\beta}(a_k)Y_{\alpha-\beta}(\sigma a_k)=0, \quad (4.5)$$

Using the boundary conditions (4.2a),(4.2b) and (3.7), we obtain

$$\bar{v}^*=\frac{2J_{\alpha-\beta}(\sigma a_k)}{\pi J_{\alpha-\beta}(a_k)}\left[\frac{1}{s+a_k^2}-\frac{1}{s}\right]. \quad (4.6)$$

Applying Laplace and generalized finite Hankel type inversion, we obtain

$$v=\pi l^{-R/2}\sum_{k=1}^{\infty}\frac{J_{\alpha-\beta}(a_k)J_{\alpha-\beta}(\sigma a_k)}{J^2_{\alpha-\beta}(a_k)-J^2_{\alpha-\beta}(\sigma a_k)}\left[J_{\alpha-\beta}(la_k)Y_{\alpha-\beta}(\sigma a_k)-J_{\alpha-\beta}(\sigma a_k)Y_{\alpha-\beta}(la_k)\right](e^{-a_k^2t}-1), \quad (4.7)$$

If we take $\alpha=1/4+\mu/2, \beta=1/4-\mu/2$ in (4.7) then we obtain the same result as obtained by Nanda [7].Further when $R\to 0$,we get the case considered by Sneddon [8].

Remark: Author claims that the results established here in this paper are more general than that of Eldabe and et al [5].

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